GENERAL PROBLEMS

OF

SHADES AND SHADOWS.

FORMED BOTH BY PARALLEL AND BY RADIAL RAYS; AND SHOWN BOTH IN COMMON AND IN ISOMETRICAL PROJECTION: TOGETHER WITH THE THEORY OF SHADING.

BY

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PREFACE.

WHEREFORE thus pursue a shadow, some may ask, seeing the title of this volume, and the complexity of some of its figures.

We will endeavor to reply.

The study of Shades and Shadows is an application of the general problems of Descriptive Geometry, in connexion with a few physical principles; and, of Descriptive Geometry, no one, who has occasion to be conversant with forms, singly or in combination, can know too much, either for practical purposes, or as a promoter, in its peculiar way, of mental power.

In particular, having in a previous work on the General Problems of Descriptive Geometry, disposed of the Projections, Intersections, Tangencies, and Developments of geometrical forms in the abstract, the way is prepared for the extended direct application of the *second* and *third* of these operations, involving incidentally that of the *first*, to the concrete geometrical subject of Shades and Shadows. For a Shade is always bounded by a line of *contact*, and a Shadow by one of *intersection;* and before either can be found, the body casting, and the surface receiving the shadow, must be given by their *projections.*

The easy *logical sequence* of "General Descriptive" Problems, and "Shades and Shadows" as a theoretical branch of applied "Descriptive," is, therefore, one point of interest attaching to the study of the latter subject.

Further, there is *beauty* in the idea. that for any distance and direction of the source of light, and for any form and position of the bodies casting and receiving shadows, the mind can know, and the hand can execute those shadows truly.

The *utility*, however, of delineated shades and shadows, in rendering working drawings at once not only more beautiful, but more intelligible, because more nearly conformed to reality, is the chief ground of inte-

rest in the study of them. The student, therefore, who has experienced any degree of pleasure in the pursuit of the general problems of "Descriptive," may, without probable disappointment, promise himself added enjoyment in the study here introduced to him.

Once more: The many interesting theoretical and practical particulars in which the science of optics is found related to the special study of shades and shadows, lend interest to the latter, as affording a field for observation and reflection to be enjoyed in common by the artist, the physicist, and the geometrical draftsman. Indeed, it is hoped that Book II. of this volume, especially Chapter II., on the execution of shades and shadows, will be found worthy of the attention of artists.

A word now as to the plan of this work.

Great care has been bestowed on the statements of general principles, which, in any effective study of the subject, must be mastered as fully as the problems.

Again: with his mind properly guarded against perplexity, by due comprehension of a simple, and manifestly rational, classification of the problems of a proposed course, the student will gain little effective assistance from the study of a protracted series of similar problems.

One good characteristic problem, under each of the heads pointed out in the preceding general table, would therefore be, at least in theory, sufficient.

The student having mastered these representative problems, would then readily refer any new problem, met by him, to its appropriate group; and would then solve it by an intelligent use of the general principles applicable to all problems of that group.

But it is usually difficult for the beginner to dispossess himself entirely of the idea that problems, which differ considerably in outward form, do not differ likewise in essential principle, and method of solution. Besides, there are often indirect and special methods of solution which really need separate exemplification; hence, a liberal number of special problems, some of them of quite a practical cast, have been scattered through this work.

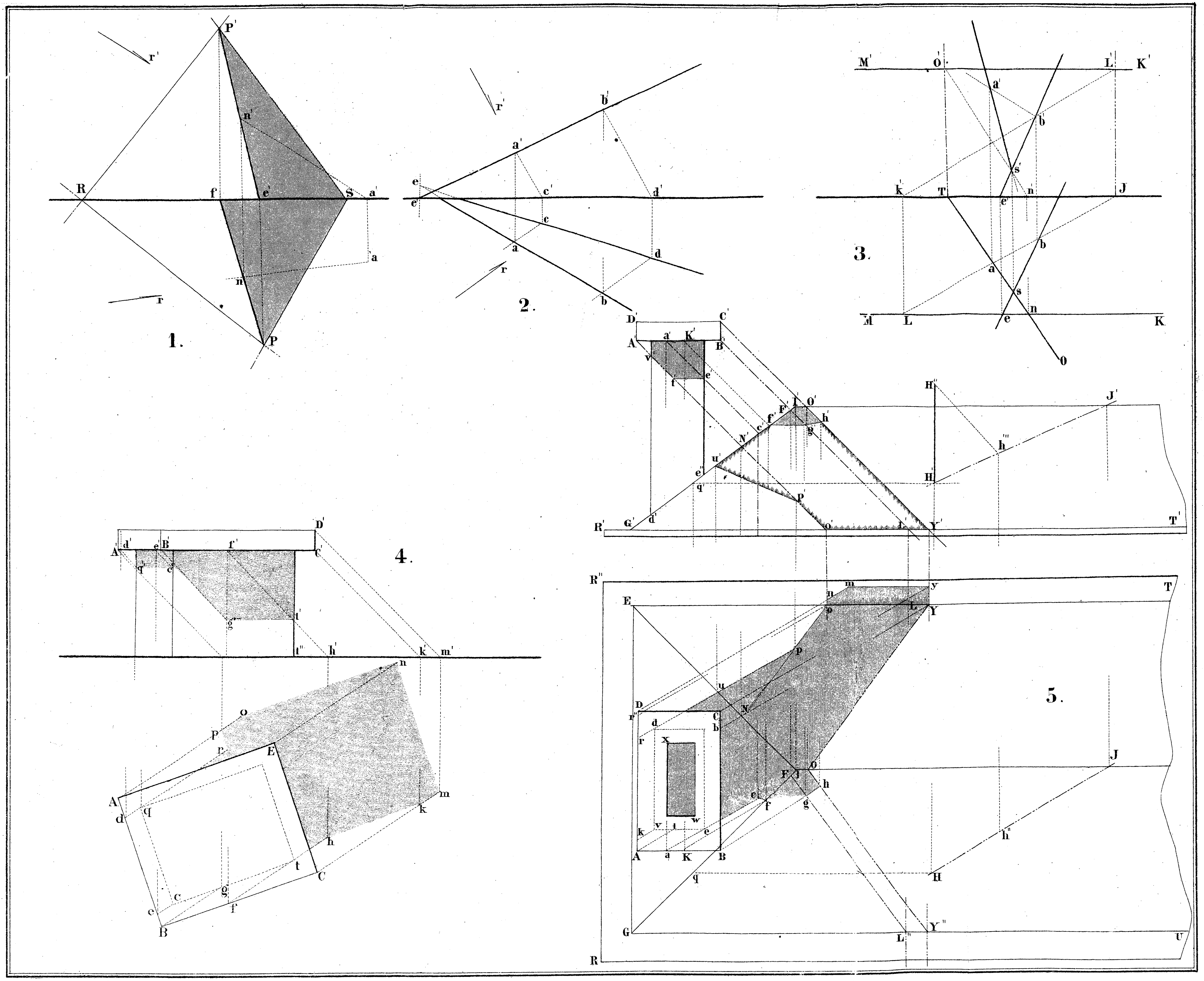
With this modification of what was at first pointed out as a theoretically sufficient course, it is hoped that the present effort will be acceptable to professors and artists, and to classes in scientific collegiate courses, and in the Polytechnic Schools; especially to those who have already become familiar with the rudiments of Shades and Shadows, as

presented in the author's "Elementary Projection Drawing," or in similar works.

As to *methods of study*, none is better, after reading the text till it is fully understood, than *repeated rehearsals* of the problems, and principles, by the student, with the book closed; the figures having been made upon a slate or portable blackboard, such as every student should have, and in any form which will truly represent the essential operations of the problems, without regard to the particular forms of the figures in the book.

The *recitations* may properly consist of mingled interrogation upon the principles, and explanations of the problems; the latter, both from the figures in the book, and from blackboard constructions; also from the finished constructions on the student's plates, when time enough is devoted in the class-room to their construction, to make due interrogation upon them possible. It will, however, probably appear on trial that no test equals the regular class recitation, in that thoroughness which is due to the student in his training.

It is a peculiar pleasure to add, that this expensive volume now appears, through the every way timely and pleasant kindness of Students of the Institute, to the Author, and, as he truly hopes, no less to themselves, in making up a liberal subscription in aid of its hitherto delayed publication.



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SHADES AND SHADOWS.

Book I.

CONSTRUCTION OF SHADES AND SHADOWS.

PART I.

IN COMMON PROJECTION.

SERIES I.

SHADES AND SHADOWS FORMED BY PARALLEL RAYS.

GENERAL PRINCIPLES.

§ 1.—*Definitions, and Classification of Problems.*

1. Light proceeds from its source in uninterrupted straight lines, called rays, except when diverted by reflection or refraction, or arrested by opaque bodies; as is sufficiently proved by the impossibility of direct vision through a bent tube.

2. If the source of light be a point, the volume of rays proceeding from it, and intercepted by an opaque body, will be a cone, having the luminous point for its vertex. *If this point be at an infinite distance from the opaque body*, the cone becomes a cylinder, and *the rays of light will be sensibly parallel.* The latter case is the one considered in the present series of problems, and is substantially the same as that of bodies illuminated by the Sun, for, on account of his great distance from the Earth, *the solar rays to all points of any terrestrial object, are sensibly parallel.*

3. Let there now be given, in the following order of successive position, a source of light, an opaque point, and an opaque surface. The opaque point will intercept one of the rays from the source of light. The path of the ray, beyond the opaque point,

will therefore be a line of darkness, and its intersection with the opaque surface will be a dark point.

The line of darkness is the *shadow in space* of the given point; the dark point on the opaque surface is the *superficial* shadow, or, simply, the shadow of the same point.

4. The shadow of a *point* upon any surface is, therefore, the intersection of a ray (meaning the indefinite path of the ray) drawn through the point, with the given surface.

5. Any plane containing a ray, is a plane of rays; for any number of rays can be drawn in the plane, and parallel to the given ray, and these rays will, collectively, constitute a plane of rays. Hence a plane of rays can be drawn through a straight line having any position in space. For the plane containing this line, and a ray through any point of it, will be the required plane of rays. Furthermore, since two lines determine a plane, but *one* plane of rays can be passed through a given line, unless the line be parallel to the rays of light.

6. A line being made up of points, its shadow will consist of the shadows of all its points. Rays of light being parallel, the rays through all the points of a straight line, will form a *plane of rays* (5), unless the line coincides with a ray. The shadow of a *straight line* is, therefore, in general, the line of intersection of a plane of rays passed through it, with the surface receiving the shadow. When the line coincides with a ray, its shadow is its own point of intersection with the given surface.

When a line is *parallel to any plane*, its shadow on that plane is parallel to the line itself.

The shadows of *parallel lines, on the same plane*, are parallel. Also the shadows of the *same line on parallel planes* are parallel; only one of them, however, can be real.

7. The parallel rays through all points of a curve, will form a *cylinder of rays*, unless the curve be plane, and coincides with a plane of rays. Hence the shadow of a *curved line* is, in general, the intersection of the cylinder of rays having the curve for its directrix, with the given opaque surface. When the curve is plane, and in a plane of rays, its shadow is the line in which that plane intersects the opaque surface.

8. The shadow of a *surface* is a surface, unless it is a plane surface and coincides with a plane of rays, when its shadow will be a line, as in the case of a plane curve lying in a plane of rays.

9. The shadow of a *solid* is always a surface, which is known when its boundary, called its *line of shadow*, is found. Let the given body be circumscribed by parallel tangent rays. These rays will form a circumscribing *tangent cylinder of rays*. The curve of contact of this cylinder and the body, is evidently the boundary between the side of the body towards the source of light, and the part which is opposed to the light. The former is the illuminated part of the body, the latter is the *shade of the body*, and the line of contact is called the *line of shade*.

10. Therefore 1*st*. From all the portion of space within the tangent cylinder of rays, and beyond the given body, light is excluded, and that portion is called the *shadow in space*, of the body. 2*nd*. The area, bounded by the intersection of the cylinder of rays with the surface receiving the shadow, is the *shadow of the body* on that surface. 3*d*. The line of shadow is thus the shadow of the line of shade. 4*th*. The line of shade on a body must therefore be known, before its line of shadow can be found.

11. Any *one point* of a curve of shade upon a body, is the point of contact of one element of the tangent cylinder of rays; that is of one ray: and the intersection of *that* ray, with any surface beyond the given body, is the *shadow of that one point* of the curve of shade. Both shades and shadows are found, in practice, according to this statement, *i.e.* by separately finding single points and joining them.

12. The constitution, and mode of construction, of the line of shade of a body, will depend on the nature of the surface of that body. The line of shade on a *plane sided* body, is composed of those edges, collectively, which divide its light, from its shaded surfaces. On a *developable single curved* surface, it consists of the elements of contact of tangent planes of rays. On a *warped* surface, it consists of a succession of points, one on each of a succession of elements, these points being the points of tangency of planes of rays. On a *double curved* surface, it consists strictly of the curve of contact of a circumscribing tangent cylinder of rays.

The *problems of shades* may therefore be arranged in the order of the four kinds of surfaces as just named.

13. Every problem of shadows, evidently reduces to finding the shadow of a point on a given surface; that is, to finding the intersection of a straight line (ray) with the surface. The method of finding this intersection depends on the nature of the

given surface, hence the *problems of shadows* may be arranged in the same way as those of shades may be.

But the construction of the shadows of straight lines being generally a little simpler than that of the shadows of curves, the surface receiving the shadow being the same in both cases, the former may be made introductory to the latter, in each of the four principal groups of problems of shadows.

14. Rehearsing the principal conclusions thus far reached, we have the following:

The *curve of shade* of a body is the *curve of contact* of a *tangent cylinder of rays*, with that body. The *curve of shadow* of the same body, is the *curve of intersection* of the same cylinder with the surface receiving the shadow.

Any *one point of the curve of shade*, is the *point of contact* of *one element of the cylinder*, and this element is a *ray of light*. The *intersection* of the *same ray*, or *element*, with the surface receiving the shadow, is the *shadow of this same point* in the curve of shade.

15. Here, expanding the last article somewhat, we observe that the line of shade is either straight or curved, hence the surface of rays passed through it—and whose intersection with any other surface is the line of shadow—will always be a plane when the line of shade is straight, and single curved when the line of shade is a curve. In the latter case the line of shade will be the directrix of the single curved surface of rays. When the rays are parallel, the single curved surface of rays will be cylindrical; when they are diverging, it will be conical, having the luminous point for its vertex. Hence, for all combinations of straight and curved lines of shade, with parallel rays, and rays diverging from a point, the curve of shadow—defined as found by the problems of orthographic projection—will be *the intersection of a plane, a cylinder, or a cone, with the surface receiving the shadow.*

16. The case, in which the source of light is a luminous *point*, either at a finite or infinite distance, is the only one which needs to be considered. For, by way of a simple example of simultaneous illumination from several points, let a sphere be exposed to the light from a finite luminous straight line. Each point of such a line would be the vertex of a separate tangent cone of rays, whose circle of contact with the sphere would be the circle of shade due to that cone. Hence, only so much of the sphere as received no light from either extremity of the luminous line, would be totally in shade; and only so much as received light

from every point of the line would be totally illuminated. The intermediate portion would be in partial shade, which would become more and more complete as the total shade should be approached. In order, therefore, to secure both *simplicity and precision* in the diagrams of shades and shadows, the source of light is assumed to be a *point*.

§ 2.—*Graphical Representation of Shades and Shadows.*

17. In passing to the graphical construction of shadows, it is to be observed, that, in representing them in orthographic projection, the planes of projection, the projecting lines, and the use of these, are similar to the same things, and their use, as found in any other problems of orthographic projection.

18. The three things given in space (3) for the production of a real shadow, are given in projection, in representing that shadow; only, as the source of light is supposed to be indefinitely remote, it is only virtually represented, by the projections of the parallel rays proceeding from it.

If the source of light were a near luminous point, its projections would be given, together with rays proceeding from it in diverging lines.

19. The ordinary rules of orthographic projection, concerning given and auxiliary, hidden and visible lines and planes, are observed in problems of shades and shadows. Moreover, visible shades or shadows are represented, either by a tint of indian ink or by parallel lines. Hidden shades or shadows are represented only by their boundaries, which are made in dotted lines; plain, or fringed, as in Pl. I., Fig. 5, for greater distinctness.

20. In constructing shadows merely as a *geometrical exercise*, the main object is to indicate clearly their position. Neatness and distinctness are, therefore, all that is required in their execution; hence, to save time, they are represented in flat or uniform tints. In making finished drawings, the shades and shadows are done in graduated tints, according to their actual appearance.

21. In industrial applications of problems of shades and shadows, the light is usually taken to correspond with the body diagonal of a cube, two of whose faces coincide with the planes of projection. That is, the light coming from the left, forward and downward, makes an angle of 35° 16′ with the planes of

projection, and its projections make angles of 45° with the ground line.

In the following general problems, the light is taken indifferently in any direction, since the general methods of solution are not altered by so doing.

22. Methods of solution are distinguished as *General* and *Special*, also as *Direct* and *Indirect*.

General methods consist of such operations as apply equally well, in theory, though not always with equal practical convenience, to whole classes of problems.

Special methods are such as are *founded on the distinguishing peculiarities of individual problems*, or groups of problems. They are therefore as numerous as those peculiarities, and hence afford the natural field for the exercise of ingenuity in their discovery and application.

Direct methods are those which immediately and obviously conform to the usual definition of a shade or a shadow; as when a point of shade is found as the point of contact of a *given* ray with a *given* surface.

Indirect methods are those in which auxiliary magnitudes are employed, to avoid the laborious constructions which direct methods may sometimes occasion; as when we find a point of shadow on some irregular surface, by noting where the easily found shadow upon some secant plane meets the intersection of that plane with the surface.

SERIES I.

SHADES AND SHADOWS FORMED BY PARALLEL RAYS.

DIVISION I.

SHADES AND SHADOWS ON RULED SURFACES.

23. The *line of shade* on any ruled surface is obtained by finding, either by inspection, or by construction—by means of tangent rays or planes of rays—the edges, elements, or points which constitute that line of shade.

24. A *shadow* cast by a body upon any ruled surface, is determined by finding an *element* of that surface, and a *point* of the line of shade of the body, both of which shall be in the same plane of rays. A ray through the *point* of shade will then intersect the *element* of the ruled surface in a point of the required shadow.

CLASS I.

SHADES AND SHADOWS ON PLANE SURFACES.

§ I.—Shades.

25. The shade of an opaque *plane*, or *plane figure*, is the whole of one side of such plane or plane figure. The former being of indefinite extent, can have no line of shade. That of the latter is its perimeter.

26. The line of shade of any *plane-sided body*, as a pyramid, consists of those edges, collectively, which divide its illuminated from its unilluminated surfaces (12). This line of shade can often be determined by inspection, having given the relative position of the body and the rays of light.

The two following problems will show the method of finding lines of shade on plane-sided bodies, when they cannot be determined by inspection.

Problem I.

To determine, first, whether the line of intersection of two oblique planes is, or is not, a line of shade; and, second, to determine, by inspection, the compound line of shade on a given plane-sided body.

First:—Principles.—After finding the projections of the line of intersection of the given planes, by Prob. V., Des. Geom., assume any point on that line, and draw the projections of a ray of light through it. If the ray, thus given, pierces either plane of projection between the traces of the given planes on that plane, it shows that both planes are illuminated, both being pierced by rays, and hence that their intersection is not a line of shade. On the contrary, if the ray pierces either plane of projection outside of the traces of the given planes on that plane, it shows that the intersection of the given planes is a line of shade.

Construction.—Pl. I., Fig. 1. Let RS be the ground line, PRP′ one of the given planes, PSP′ the other, and Pf—P′e′ their line of intersection. n, n' is an assumed point of this intersection, through which the ray, na—$n'a'$, parallel to the given ray, r—r', is drawn. This ray pierces the horizontal plane at a, a', indicating, according to the principles of this solution, that Pf—P′e′ is a line of shade.

Remarks.—*a.* Hence the projections of the visible portion of the unilluminated plane, PSP′, are shaded, and Pf—P′e′ is inked as a heavy line.

b. The above construction is applicable to the case of any pyramid, in a simple, or oblique position, when we wish to determine its line of shade. In such cases, unless the pyramid rests upon a given plane, it will first be necessary to determine the traces of its faces on some plane, whose intersections with the rays through points in the edges of those faces will also be constructed.

27. As a second illustration of this problem, see Pl. IV., Fig. 14, where PQP′ and PRP′ are the given oblique planes. P′a′ and Pb are the projections of their intersection, and m,m' is a point on that line. mn—$m'n'$ is a ray of light through m,m', and it pierces the horizontal plane at n, outside of both planes. Hence, when Pb—P′a′ is regarded as the edge of the salient diedral angle, indicated by the portions, QP and RP, of the traces, it is an edge of shade for the direction, mn—$m'n'$ of the light.

If PQP′ as now shown, and that part only of PRP′, extending in front of the edge $Pb - P'a'$, are considered as the surfaces exposed to the light, the left side of each plane is then the one considered, and both of these surfaces are in the light.

The next problem affords a more practical example of the general constructions just given.

28. *Second.—The lines of shade on the abacus and pillar.*—See Pl. I., Fig. 4, where the projections of a square pillar and its abacus, resting on the horizontal plane, are given, together with the direction of the light, Cm—$D'm'$.

Here it is evident that $t - t''t'$; C — C′D′; the upper edges CE and EA; the vertical edges at A and q; AB — A′B′, and BC — B′C′, are the edges which constitute the compound line of shade of the given body. They are therefore (9—11) the edges which cast shadows on the pillar, or on other surfaces; and they are identical with the lines, which, in geometrical line-drawings, are inked as heavy lines.

Problem II.

To determine the edges of shade on a pyramid, whose axis is oblique to both planes of projection.

Construction.—Pl. IV., Fig. 15. To make the problem quite definite, let G″L″ be the ground line of an auxiliary vertical plane, parallel to the axis of the pyramid, which we will suppose to be a triangular one, in order to avoid needless repetition of the same operations of construction. Let the plane of the base be perpendicular to the axis, and let $a''c''$ be its trace on the auxiliary vertical plane. Let ABC be the true size and form of this base, seen when its plane is revolved about $a''c''$, as its trace, into the auxiliary vertical plane, and let D be the projection of the axis on the plane of the base. Then by counter-revolution, A, B and C return in arcs, projected in Aa'', etc., perpendicular to $a''c''$, to the points a'', b'' and c''; and D, the foot of the axis, to d''. Then make $d''v''$, perpendicular to $a''c''$, equal to the axis of the pyramid, and join v'' with a'', b'' and c'' to complete the auxiliary projection.

To make the horizontal projection, make aa'', etc., equal to Aa'', etc., and perpendicular to the ground line G″L″; and v —

abc will be the horizontal projection of the pyramid. The vertical projection, $v' - a'b'c'$, is found, as in all similar cases, by projecting up $v - b$, etc., in perpendiculars to GL, and by making the heights of the points v', $- b'$, etc., above the ground line GL, equal to their heights $v''k$, $b''h$, etc. above G″L″.

Now, to test any edge of the pyramid, to see if it be an edge of shade, or not.

First Method.—Let the edge $vb - v'b'$ be taken. Find the traces, on one plane of projection, of the planes of the faces of which the given edge is the intersection. Then draw a ray through any point of that edge, and if it meets the plane of the traces outside of those traces, when the edge belongs to a salient angle turned towards the light, the given edge is a line of shade. Accordingly, $vb - v'b'$ pierces the horizontal plane at m, and $cb - c'b'$ pierces it at n, giving mn, as the horizontal trace of the plane of the face $vbc - v'b'c'$. Again, $va - v'a'$ pierces the horizontal plane at a, giving ma as the horizontal trace of the plane of the face $vab - v'a'b'$. Now a ray, $bp - b'p'$, through any point bb' of the intersection, $vm - v'm'$, of these planes, pierces the horizontal plane at p, outside of both traces, ma and mn. But the parts of these planes which are the faces of the pyramid, estimated from the common edge, $vb - v'b'$, have ma and mO for their horizontal traces. Hence the direction of the light evidently strikes only the interior face of each of these planes, that is, the face towards the axis of the pyramid; so that the exterior surfaces vbc and vba are both in the dark, and $vb - v'b'$ is not a line of shade.

If now we find O, the intersection of the edge, $vc - v'c'$ with the horizontal plane, aO is the horizontal trace of the plane of the face $vca - v'c'a'$, and vcO is a salient edge, similar to Pb — P$'a'$ in Fig. 14, so that the ray, $cr - c'r'$, by piercing the horizontal plane outside the traces On and Oa, shows that $vc - v'c'$ is an edge of shade, which, indeed, was known as soon as it was found that $vb - v'b'$ was not such an edge.

Second Method.—Find the shadows of all the edges of the pyramid on either plane of projection. Then, if, for example, the shadow of $vc - v'c'$ upon the horizontal plane is further from the pyramid, in the direction of the light, than that of $vb - v'b'$, it would show that $vc - v'c'$, only, of those two edges, really cast a shadow. Hence it would be shown that $vc - v'c'$ was an edge of shade (26). That is, we have this principle: *Those*

edges, whose shadows are the boundaries of the real shadow of a body, are the edges of shade of that body.

Remark.—The next problem, if not the preliminary general principles already given, will enable the student to construct this second method.

§ II. Shadows.

29. Here observe—*First:* That whenever a plane is perpendicular to either plane of projection, its entire surface, and all contained in it, is projected in the trace of the plane on that plane of projection. *Second:* Having a point, *p*, which casts a shadow on a plane, the projections of that shadow must be in the projections of the ray (4) through *p*.

30. From the two foregoing principles we have the following simple *general method* (22) of finding points of shadow on the planes of projection, or on any planes which are perpendicular to them. *Find, by inspection, the intersection of the linear projection of the plane receiving the shadow, with the corresponding projection of the ray through any point casting a shadow upon it. This will be one projection of the point of shadow sought. Its other projection will be the intersection of the projecting line of the point found, with the other projection of the same ray.*

It is only necessary here to remember, that the linear projection of both planes of projection, is the *ground line.*

Problem III.

To find the shadow of a straight line, on the horizontal plane of projection.

Principles.—The shadow of a straight line on a plane, is the intersection of that plane with a plane of rays passed through the given line (5–6). This shadow will therefore be a straight line. But two points determine a straight line, and two parallel lines determine a plane. Hence if, through any two points of the ggien line, we pass rays, their intersection with the given plane will determine the shadow of the given line. Also, the point in which the given line meets the given plane is a point of the required shadow, for the line and its shadow are in the same plane (of rays) and therefore intersect.

Construction.—Pl. I., Fig. 2. Let $ab - a'b'$ be the given line, and $r - r'$ the direction of the light. By (30) the ray $ac - a'c'$, drawn through any assumed point, as a, a' of the given line, pierces the horizontal plane in the point whose vertical projection is c', and whose horizontal projection is therefore c. Likewise the ray $bd - b'd'$ pierces the horizontal plane at d',d (naming that projection first which is found first). Hence cd is the shadow of $ab - a'b'$ on the horizontal plane. Also e,e', the point in which $ab - a'b'$ pierces the horizontal plane, is a point of its shadow on that plane, and hence will be found in dc produced.

Remark.—In taking a strict view of the principles of this solution, and in similar cases, it should be observed that the rays, as $ab - a'b'$, are to be regarded as cut from the plane of rays (6) passed through the line, by secant planes of rays, which may have any position, but are most readily conceived of as perpendicular to one of the planes of projection.

31. *When a line is perpendicular to a plane, its shadow, on that plane, is in the direction of the projection of the light on that plane;* for the plane containing the line and a ray through any point of it, is a plane of rays (5), perpendicular to the given plane, and is therefore a *projecting plane* of all rays intersecting the line. Hence its trace on the given plane, is both the shadow of the line (6), and the projection of a ray, upon that plane.

This principle is of very frequent application to the shadows of vertical lines upon the horizontal plane, and of co-vertical lines (that is, of lines perpendicular to the vertical plane) upon the vertical plane.

Problem IV.

To construct the shadow of a square abacus upon a square pillar, and of both, on the horizontal plane of projection.

[There being no new principles in this problem, we pass at once to its graphical solution as a new illustration of the method of (30) referring to (28) for the line of shade.]

Construction.—Pl. I., Fig. 4. Let the body be seen obliquely as shown on the vertical plane, and let $Cm - D'm'$ be the direction of the light. The point d,d', found by drawing the ray qd, and projecting d at d', casts the point of shadow q,q' on the ver-

tical edge at q; q' being found at the intersection of this edge with the vertical projection, $d'q'$, of the ray through d,d'. From q' the shadow $q'c'$ is parallel to the line $de-d'e'$, which casts it, since that line is parallel to the vertical face, qc, of the pillar (6). The ray through B,B′ pierces the front of the pillar at g, whose vertical projection is in the vertical projection, $B'g'$, of the ray, at g'. Hence $c'g'$ is the shadow of the portion, $eB-e'B'$, of the edge, AB—A′B′, of the abacus, upon the front of the pillar. $g't'$, parallel to B′C′, is the shadow of the portion $Bf-B'f'$ of the edge BC—B′C′, on the parallel front of the pillar. The ray through f,f', passing beyond the pillar, pierces the horizontal plane (30) at h', h, and ht is the shadow of the portion, $t-t't''$, of the edge of the pillar, which casts a shadow on the horizontal plane. The portion $fC-f'C'$ of BC—B′C′ casts the shadow hk, beginning at h and limited at k',k by the ray $Ck-C'k'$. The edge C—C′D′ casts the shadow km, limited by the intersection of the rays $Ck-C'k'$ and $Cm-D'm'$ with the horizontal plane. Similarly, CE has mn, equal and parallel to itself, for its shadow; EA has, likewise, no for its shadow; the vertical edge at A determines the shadow op, found as km was; pr is the shadow of $Ad-A'd'$, and rq is the shadow of the vertical edge of the pillar at q.

Problem V.

Having the projections of a ray of light, upon two vertical planes at right angles to each other; to find its projections on those planes, after revolving it 90° about a vertical axis.

Principles.—When we view any object in reference to a given vertical plane, the direction of the light is regarded as fixed; and it is therefore fixed in projection, while we are finding the shadow of the object on that plane. When, however, we turn and face the object in a different direction, the light may be supposed either to turn with us, so as to come in the *same relative direction* with respect to the new projection that it did with respect to the old; or it may be supposed to remain absolutely fixed, as it is practically in Nature for any short interval. A little reflection on the latter supposition, will show that, in certain aspects of an object, only its shade could be seen; hence it is sometimes, if not generally, desirable to adopt the first view, viz. that the light turns with us, so as to come in the *same*

relative direction with respect to each new vertical plane of projection. This first view is adopted in the following—

Construction.—See Pl. II., Fig. 6. Here let the given projections of the light, on the two vertical planes, be R′L′ and R″L″, and let qpx' indicate the real position of the vertical plane of the left hand projection. By an obvious construction, we find RL, the horizontal projection of the given ray, R″L″—R′L′. Then, as the observer turns 90° to face the plane qp, the light turns with him, an equal amount, about any vertical axis, as at L, and so appears in horizontal projection, as at rL, perpendicular to RL. The vertical projections of rL are, obviously, r'L′ and r''L″.

We see now, by inspection, that, as the vertical planes, and also the two positions of the light, as seen at RL and rL, are at right angles to each other, $pt = sm$, and $pq = mn$. That is, $x'y'$ being the same for all the vertical projections of the ray, the new vertical projections, r'L′ and r''L″, make the same angle with the ground line as R′L′ and R″L″ do. So that, in practice, it is only necessary to draw r'L′ parallel to R″L″, and r''L″ equally inclined with R′L′,—but in an opposite direction—in order to obtain the projections of the light as it proceeds in viewing the side or left hand elevation.

When the projections, rl—$r'l'$, of the light make the conventional angle of 45° with the ground line, as in Pl. II., Fig. 7, the vertical projection, $r''''l''''$, of it on the right hand plane, as it comes when the observer has turned, from viewing that plane, to view the left hand one, simply inclines at 45° to the ground line in the opposite sense, from $r'l'$.

Problem VI.

To find the projections of the shadows on the parts of a timber framing, shown in two elevations, on two vertical planes at right angles to each other.

Principles.—The chief new principle in this problem is, that a point of shadow may exist, geometrically, on a material surface produced; though, physically, only on the real portion of that surface.

Construction.—Pl. II., Fig. 8. MJ—J′J″ is a horizontal timber, resting on the transverse timber AJ—A′B′C′, and on the lateral braces kh—$g'y$ and kh—nu. These braces are

framed into the post LH – A′H′, in front of which is the front brace GP—G′P′. The left hand elevation is supposed to be made on a vertical plane at right angles to the plane of the paper. Taking the light in this practical problem in the conventional direction (21) GL – G′L′ are the projections of a ray, as it proceeds in viewing the object as seen on the right hand plane of projection.

This being established, we are prepared to construct the shadows in Fig. 8. The point A,A′ casts a shadow on the front brace at *a*, whose other projection, *a*′, is the intersection of the projecting line, *a*—*a*′, with the other projection, A′*a*′, of the ray A*a*. A – A′B′, being parallel to the face of this brace, its shadow, *a*′*a*″, is parallel to A—A′B′. Next, *c*,*c*′ and L,L′, the shadows of the points *a*,*a*″ and G,G′ in the front right hand edge of the brace, are found precisely as *a*,*a*′ was, and they determine L′*c*′, the position of the imaginary shadow (in this figure) of this edge on the plane of the front of the post produced. The shadow of the back left hand edge, which pierces the post at P,P′, begins, therefore, at P,P′ and is parallel to L′*c*′, as seen at P′K. The front of the lateral brace, *g*′*y*, being parallel to the front of the post, the shadow of G*a* — G′*a*″ upon it, will be parallel to L′*c*′. One point of this shadow is *b*,*b*″, found by drawing the ray *ab*—*a*″*b*″; hence *b*″*f*, parallel to L′*c*′, is the required shadow. The point A, B′ casts the shadow *b*,*b*′, found like the preceding, on the plane of the front of the lateral brace. Then *b*′*e*′, parallel to B′C′, and limited by the ray through C,C′, is the shadow of AC—B′C′ on the same plane. The portion of shadow on the brace, bounded by C′*e*′, is cast by the edge, CJ—C′, of the transverse timber, and *b*′*b*″ is the shadow of A—A′B′ on the plane of the front of this brace. Finally, for the right hand elevation, the edge J—J′J″ casts the shadow *n*—*y* on the lateral braces, found by projecting over *k*″, where a ray through any point of this edge meets the plane of the front of these braces.

Proceeding to the side elevation, two planes of rays, containing the edges whose projections are J and M, will include between them the shadow in space (10) of the longitudinal timber JM—J′J″. Their traces on the side of the transverse timber, will therefore bound the shadow on that timber. A portion of one of these traces is seen at J*k*″. The edge AB—B′ casts a shadow on the side of the front brace at T′, found by drawing the ray B′T′ (Prob. V.). According to the construction of all the other points of shadow in this problem, T, its other projection, is at

the intersection of the projecting line, T′T, with the other projection, AT, of the ray through A, B′, as it comes—according to previous explanation (Prob. V.)—when viewing the framing as seen in the left hand projection. AB being parallel to the side of the front brace, its shadow, *tv*, through T, is parallel to itself. Lastly, the brace *hk″—g′y* casts a shadow on the side of the post. A little consideration of a model will show, that, when the brace makes a *less* angle with the horizontal plane than the light does, its *left* hand *front* edge and *right* hand *back* edge, as seen on the left hand elevation, will cast shadows. When the former angle is the greater of the two, the *right* hand *front* edge and *left* hand *back* edge will cast shadows. When the two above angles are equal, the front and back planes of the brace will be planes of rays, and their traces, *kk‴* and *gh*, produced, will bound the shadow of the brace.

In the present problem, the brace makes a less angle with the horizontal plane than the light does, hence the edges, *k′y—kk″* and *g′f—hp*, cast shadows on the post. The shadow of the former edge begins at *k*, and of the latter at *h*, these being the points where these edges pierce the post. Assume any point F, F′ and draw the ray F′H′—FH, which, according to the previous constructions, pierces the plane of the side of the post at H′,H. H*k* is therefore the shadow of *kk″*; and *sh* parallel to it is the shadow of *hp*.

Problem VII.

To construct the shadow of a circle on the vertical plane of projection.

Principles.—The only essential difference between this and the previous problems is, that, in case of a curve, a plane of rays can generally contain but one or two points of it, while it may contain the whole of a straight line. But we have seen that the shadow of a straight line is practically determined by the shadows of two of its points. Now the shadow of a point is found in the same way, whether that point be on a straight line or on a curve; hence shadows of straight lines, and shadows of curved lines, are not here distinguished as belonging to distinct classes of problems.

According, therefore, to the method of (30), again, we have the following—

Construction.—Pl. III., Fig. 10. Let the plane of the circle be perpendicular to the ground line. Then the equal lines, AB and C′D′, perpendicular to the ground line, will be the projections of the circle. Let Aa—A′a' be the projections of a ray of light. This ray meets the vertical plane at a', the shadow, therefore, of A, A′, the foremost point of the horizontal diameter of the circle. The ray Bb—A′b' determines the shadow of the point B, A′, at b'; the rays Cc—C′c' and Cc—D′d' determine the points of shadow, c' and d', of the extremities of the vertical diameter C—D′C′. Assuming either projection, as E, of any other two points on the circle, their other projections must be constructed by some one of the methods of (Prob. XL., Des. Geom.), E is the horizontal projection, as indicated by the construction, of the two points E′ and F′. Each of these, again, is the vertical projection of two points, of which the two foremost, one in each pair, are horizontally projected at G—found by making CG = CE. These four points, G, E′; G, F′; E, E′; and E, F′ cast shadows at e', g', n', and f', respectively. By joining the points thus found, and tinting the inclosed figure, we shall have the required shadow of the circle on the vertical plane.

Remarks.—*a.* The lines a—a'; b—b'; C′—c', and D′—d', are tangents to the shadow, which is an ellipse. Also, a'—b', and c'—d', are conjugate diameters, from which the axes can be found, if desired, as commonly shown among problems on the ellipse.

b. Observing that in Pl. I., Fig. 4, km is the shadow of a vertical edge at C on the horizontal plane of projection, and that in Plate III., Fig. 10, $a'b'$ is the shadow of the diameter AB—A′ on the vertical plane, we conclude from inspection, what was proved in (31), that *whenever a line is perpendicular to either plane of projection, its shadow on that plane is in the direction of the projection of a ray of light on that plane.*

33. Pl. III., Fig. 11, illustrates the forms of the shadows of a circle, whose plane is oblique to the direction of the light, upon several planes variously inclined to the plane of the circle.

AB represents a vertical projection of a horizontal circle, and ABGC the vertical projection of an oblique cylinder of rays having the circle AB for its base, and parallel to the vertical plane of projection. The traces drawn through C, are the traces of a number of planes, perpendicular to the vertical plane

of projection, and whose intersections with the cylinder of rays are the shadows of the circle AB.

DC is a circular shadow, equal to AB, its plane, DC, being parallel to AB. CF is an elliptical shadow less than the circle AB, and whose transverse axis, projected at the point *f*, is equal to the diameter of AB. CF being a plane of right section, contains the least shadow of AB. CH is an elliptical shadow greater than AB, CH, its transverse axis being greater than AB, and *h*, its conjugate axis, being equal to AB. Between CF, the least, and CH, the greater shadow, there must be one equal to the circle AB. This is found by making GCF=FCD when CG=CD, and as all the diameters *h*, *g*, *f*, etc., are equal, CG is another circular shadow, called the *subcontrary section* of the cylinder, to distinguish it from the *parallel section* CD. Finally, all shadows between the circular ones CD and CG, are ellipses, less than, and all exterior to these are ellipses greater than the circle AB.

34. The foregoing problems being sufficient to illustrate the method of (30) we have the following *general method* for the more general case in which the plane receiving the shadow is not perpendicular to either plane of projection. *Pass planes of rays, each of which will cut a point,* p, *from the line* L, *casting the shadow, and a line,* k, *from the plane,* Q, *receiving the shadow. The point in which a ray through the point,* p, *meets the trace,* k, *will be a point of the shadow of* L *on* Q.

Remarks.—a. When L is a straight line, the plane of rays may always contain it, instead of merely cutting points from it, and then the trace of the plane of rays on the plane Q, will be the shadow of L.

b. The trace just mentioned, must, however, be constructed by the above method, as seen in the line *cd*, Prob. III. Hence, practically, planes of rays are not *immediately* passed through straight lines; i. e. without construction, unless those lines are perpendicular to a plane of projection; in which case the planes of rays containing them will be so also, and their traces will be at once known. (29.)

Problem VIII.

To construct the shadow of a straight line on a plane whose traces are parallel to the ground line.

Construction.—Pl. I., Fig. 3. *as*—*a's'* is the given line, and MK and M'K' are the traces of the given plane. OTO' is a vertical auxiliary plane which contains the given line, *as*—*a's'*, and intersects the given plane in the line *n*T—*n'*O'. Hence, from Des. Geom. (Prob. VI.), the given line intersects the given plane at *s'*,*s*, which (Prob. III.) is one point of the required shadow. LJL' is another vertical auxiliary plane, containing the ray *ab*—*a'b'*, drawn through the point *a*,*a'* of the given line *as*—*a's'*. This plane intersects the given plane in the line LJ—*k'*L', hence the ray through *a*,*a'* intersects the given plane at *b'*,*b*, which by (34) is therefore another point of the required shadow. Hence *bs*—*b's'* is the required shadow.

Remarks.—*a.* The auxiliary planes might have been placed perpendicular to the vertical plane of projection. In that case, the horizontal projections *b* and *s* of the points of the shadow would have been found first.

b. Let this problem be solved thus, and also when the given plane has any oblique position.

Problem IX.

To find the shadow of a chimney, situated on the end portion of a hipped roof, upon the end and side roofs.

Principles.—It only need be noticed here, *first*, that as most of the edges of the chimney are perpendicular to one, or the other, of the planes of projection, the auxiliary planes of rays may all contain edges of the chimney, and be also perpendicular to one of the planes of projection; and *second*, that when a plane, P, is perpendicular to either plane of projection, all lines in that plane, P, will be projected on the plane of projection, to which it is perpendicular, in its trace on that plane (29).

Construction.—Pl. I., Fig. 5. URR''T—R'T' is the horizontal plane of the eave-trough, intersected by the roof in the lines

TE, EG, and GU. GIU—G′I′T′ is the front roof, EIG—G′I′ the end roof, perpendicular to the vertical plane of projection, and EIT—G′I′T′ is the back roof; *xw* is the chimney-flue, *dve* a horizontal section of its outside surface, and ABD—A′C′D′ is its abacus.

The shadow of the abacus on the front of the chimney, is found as in Prob. IV.

To find the shadows on the roof: *First*, the shadow on the front roof. C′Y′ is the vertical trace of a plane of rays through BC—C′, and perpendicular to the vertical plane of projection. According to principle *second*, O′Y′ is the vertical projection of its trace on the front roof. Y′—Y″Y is its trace on the horizontal plane of the eave-trough; hence, projecting O′ at O we have OY and OY″, for the horizontal projections of its traces on the front and back roofs. Drawing the ray B*h*—whose vertical projection is C′Y′—and projecting *h* at *h*′, the point *h*, *h*′ is the shadow of B,C′, and *h*O—*h*′O′, the shadow of BC—C′ on the front roof. Drawing a similar plane of rays, B′L′, its traces on the roof are F′L′—FL″, parallel to O′Y′—OY″, a line from L parallel to YO, and the short line, parallel to GE, across the end roof at F. The ray B*g*—B′L′ meets the trace FL″ at *g*,*g*′, giving *hg*—*h*′*g*′ for the shadow of B—B′C′. From *g*,*g*′ the shadow of AB—A′B′, on the parallel front roof, is the parallel line, *gf*—*g*′*f*′. Drawing the ray *f*K—*f*′K′, we find the portion BK—B′K′ whose shadow is *gf*—*g*′*f*′.

Second.—The shadow on the end roof. *ec* is the trace, on this roof, of a vertical plane of rays through the vertical edge, *e*—*e*″*e*′, of the chimney. This plane cuts the edge, AB—A′B′, of the abacus in the point *a*,*a*′, through which, drawing the ray *a*′*c*′—*ac*, and projecting *c*′ at *c*, we find *ec*, the shadow of the portion *e*—*e*″*e*′ of the edge of the chimney. Above *e*′ this edge is in the shadow of the abacus on the chimney. Joining *c*,*c*′ and *f*,*f*′; *cf*—*c*′*f*′ is the shadow of *a*K—*a*′K′. Likewise, *du* is the shadow of the vertical edge at *d*. By projecting *u* at *u*′, and by drawing the vertical projection of a ray through *u*′, we could find the portion of the edge from *d*,*d*′, upwards, which casts the shadow *du*—*d*′*u*′.

Third.—To find the shadow on the back roof and eaves plane. A′*o*′ is the vertical trace of a plane of rays parallel to the plane C′Y′Y″. Its trace on the back roof is N′*o*′—*o*N. It intersects the edge, *d*—*d*′*v*′, of the chimney at *d*,*v*′, through which, draw-

ing a ray, $v'o'$—dp, we find p,p' the limit of the shadow of d—$d'v'$ on the back roof. This point p,p' is also by construction the shadow of r,A′ where the ray $p'v'$—pd meets the edge AD—A′ of the abacus. Drawing the ray or'', rr'' is determined as the portion of this edge, which casts the shadow po—$p'o'$. The remaining portion, Dr'', casts the parallel shadow on—o', a part of the trace of the plane of rays A′o' on the plane of the eave-trough, and limited by the ray Dn—A′o'. Drawing the ray Yb, we find the limit of that portion of the edge BC—C′ which casts a shadow on the back roof. bC casts the equal and parallel shadow, Yy, on the parallel plane of the eave-trough. Likewise ym is equal and parallel to CD—C′D′, and is limited by the rays Cy and Dm. Then mn is the shadow of D—A′D′, which completes all the required shadows.

Remarks.—a. The boundary of the shadow on the back roof, as seen in vertical projection, is fringed, to give it greater distinctness.

b. To locate either projection of a point on the side roofs, one projection of such a point being given. Let H be the horizontal projection of a vertical rod on the front roof. Draw Hq parallel to the trace, GU, of the front roof; q is vertically projected at q', and q'H′, parallel to G′T′, is the vertical projection of qH; hence H′H″ is the vertical projection of the rod at H. This construction would be used in locating the intersection, with either front or back roof, of a chimney situated on either of those roofs. (See Des. Geom. Prob. X.)

c. To find the shadow of the rod, H—H′H″, pass a vertical plane of rays through it; HJ, parallel to Bg, is the horizontal projection of the trace of this plane on the front roof. Project J at J′ and H′J′ will be the vertical projection of the same trace. The ray H″h'''—Hh'' meets this trace at h''',h'', giving Hh''—H′h''' for the projections of the shadow of the rod. This construction could have been employed in finding all points of the shadow of the chimney, instead of making the planes of rays perpendicular to the vertical plane. Observe that in the construction given for the chimney, we find the horizontal projections of points, as h, first; while in the construction of $h''h'''$ we find h''' first.

35. In dismissing the subject of shadows on planes, it may be noticed, that it is sometimes advantageous to consider the ray of light, itself, as making an angle of 45° with one of the

planes of projection. For, suppose such a ray to be passed through the uppermost point of a vertical line, and to make an angle of 45° with the horizontal plane. Evidently, the distance from the foot of the line to the intersection of the ray with the horizontal plane—which would be the shadow of the line—would then be equal to the height of the line. Hence, without having any vertical projection of the line, its height may be known from its shadow. Or, which amounts to the same thing, the height of a point above the horizontal plane will be known by the distance of its shadow from its horizontal projection.

The same essential principle is true in more general terms. For, by knowing the direction of the light, we can find the relative lengths of vertical lines and their shadows, and this relation being known, we can find the height of all points of an object, having given only its plan and shadow. The same principles are applicable to constructions made on the vertical plane of projection. To illustrate, let us take the following example:

Problem X.

To find the shadow of a shelf and brackets upon a vertical plane.

Construction.—Pl. II., Fig. 9. Let AB, taken as the ground line, be the horizontal trace of the given vertical surface; let CD—C′D′ be the lower front edge of the shelf, and let *cb*—*c′b′* be one of the brackets.

Let E′F be the vertical projection of a ray, drawn in any direction, relative to the ground line. It is sufficient to understand that this ray makes an angle of 45° with the vertical plane. Then the shadow, *c′d′*, of the edge *cn*—*c′* will be parallel to E′F and equal to *cn*. Next, *d′e′* can be drawn, parallel to *ca*—*c′a′*, and limited by the ray *a′e′*, and the shadow of the unshaded part of *a*—*a′b′* will be the parallel line *e′k′*.

Likewise, taking any point *f*,*f′*, in the edge CD—C′D′ of the shelf, make the ray *f′h′* equal to *fg*, then *h′k′*, parallel to C′D′, will be the shadow of CD—C′D′.

Remark.—Though not necessary, the horizontal projection of the ray may be found as follows: Make GA = E′F, then will GF—E′F be the two projections of a ray which makes an angle of 45° with the vertical plane of projection.

CLASS II.

SHADES AND SHADOWS ON SINGLE CURVED SURFACES, IN GENERAL.

SECTION I.

On Developable Single Curved Surfaces.

§ 1.—Shades.

36. *The line of shade on a developable surface, consists of those rectilinear elements along which planes of rays are tangent to the surface*, since such a line would evidently separate the illuminated from the dark portion of the surface. Since the tangent planes are planes of rays, they may be regarded as parallel to a given ray, in space. Hence the *general method* for the construction of the elements of shade on a developable surface, resolves itself into the operations of (Des. Geom. Probs. 53, &c.) "*To construct a plane, parallel to a given line, and tangent to a developable single curved surface.*" Since two such planes can generally be found in any case, there will usually be two opposite elements of shade.

37. When the indefinite surface is limited by one or more plane intersections, taken as bases, the complete line of shade will consist of the elements of shade on the convex surface, together with those portions of the circumferences of the bases which divide light from dark portions of the entire body.

38. If a plane of rays be tangent to a developable surface along one of its elements, then any secant plane will cut from the surface a curve, and from the plane of rays, a line, tangent to the curve at one point of the element of shade. Moreover, if the secant plane be perpendicular to the tangent plane of rays, the line cut by it from the latter will evidently be the projection of the light upon the secant plane, since the tangent plane, when thus perpendicular to the secant plane, becomes a projecting plane of rays upon the latter plane.

Problem XI.

To construct the elements of shade on any cylinder or cone.

Constructions.—First.—To construct the element of shade on a horizontal right semi cylinder. Pl. IV., Fig. 16. Let Fb be the trace of one of two vertical planes of projection—upon the plane of the paper—taken as the second vertical plane, and at right angles to the former one. Let the former plane be perpendicular to the common axis of the cylinder, and cylindrical abacus, shown in the figure, and let the latter plane contain the axis of the cylinder. The projections will then be as in the figure. Let De—D$'e'$ be the projections of a luminous ray. De is also the trace, on the right hand plane, of a plane of rays perpendicular to that plane; e is therefore the right hand elevation of its element of contact with the cylinder, that is, of the element of shade of the cylinder. E$'e''$ is therefore the left, or linear elevation of the same element.

Likewise, the parallel tangent plane of rays at G, determines the element of shade, G—G$''$G$'$, on the abacus.

Second.—To construct the elements of shade on a cylinder whose axis is oblique to both planes of projection. Pl. III., Fig. 13. ABF—A$'$G$'$ is the lower base of the cylinder. Ibe—I$'$J$'$ is its upper base, Cb and Mg are its extreme elements, as seen in horizontal projection, and A$'$I$'$ and G$'$J$'$, the extreme elements as seen in vertical projection. The axis of the cylinder pierces the horizontal plane at K, the centre of the lower base. The ray aL—a'L$'$, through a point a,a' of the axis, pierces the same plane at L, hence KL is the horizontal trace of a plane containing the axis and a ray. From (5) such a plane is a plane of rays through the axis. The tangent planes of rays will be parallel to the one just found, hence their horizontal traces will be parallel to KL. BB$''$ and FF$''$ are these traces, and their points of tangency, B and F, with the lower base, are where their elements of contact—the elements of shade—pierce the horizontal plane. Hence Bu—B$'u'$, and Ff—F$'f'$, are the required elements of shade; one projection of each being visible.

Third.—To construct the elements of shade on a cylinder whose axis is horizontal, but oblique to the vertical plane of projection. Pl. IV., Fig. 17. In order to construct the vertical projection of a cylinder, thus seen obliquely, it is necessary to have, first, its

projection on a plane perpendicular to its axis. Then let NZH be the horizontal projection of the cylinder, and take O″C″, perpendicular to its axis, for the ground line of a new vertical plane, parallel to the base NZ. This base will then have the circle H″O″b″ for its auxiliary vertical projection, and that of the cylinder. Now, any point as Z′, in the principal elevation, is in a projecting line, ZZ′, and at a height, Z′O′, from the ground line, equal to the height of the same point, as seen at Z″, above the ground line O″C″.

Having thus found the projections of the cylinder, we may assume any two projections of a luminous ray, and find the third. Let FL—F′L′ be the assumed projections of a ray. L,L′ is projected on the auxiliary plane at L″;—L′, and L″ being at equal heights above their respective ground lines. Likewise G″ and F′, both projected from F, are at equal heights above their respective ground lines. G″L″ is then the auxiliary projection of the ray. Hence also T″H″ and M″J″, parallel to G″L″, are the traces, on the auxiliary vertical plane, of tangent planes of rays perpendicular to that plane. Hence H″—HI and J″—KJ are two projections of the two elements of shade of the cylinder. KJ is again vertically projected at K′J′ at a height, equal to J″*x*″, above the ground line C′O′, or by projecting J into the principal vertical projection of the base at J′, and drawing J′K′ parallel to the ground line. The former construction is obviously more accurate in practice.

Otherwise: omitting all the operations of the last paragraph, the ray, NO—N′O′, may be projected into the plane, NZ, of the base of the cylinder, by projecting O,O′ at *o*″,*o*‴. Then, considering that N,N′ is its own projection on the plane NZ, we have N*o*″—N′*o*‴ to represent the projections of the ray on this plane. Hence draw a line tangent to the base, and parallel to N′*o*‴, and by the principle of (38) J′ will be its point of tangency with the base N′J′Z′, and therefore a point of the element of shade J′K′.

Fourth.—To construct the elements of shade on a cone. Pl. III., Fig. 12, and Pl. V., Fig. 18.

Principles.—A plane is tangent to a cone along an element, and all the elements contain the vertex of the cone, hence a plane of rays tangent to a cone will contain the ray through its vertex. Therefore the trace of this plane on any given plane, will contain the intersection of that ray with this given plane, and,

by (38) the point of contact of this trace with the section of the cone, made by the given plane, will be a point of the element of shade.

Constructions.—In Pl. III., Fig. 12, draw a ray, VR—V′R′, from the vertex, V,V′, of the cone. It pierces the horizontal plane, which is here the plane of the cone's base, at R. Then RT is the horizontal trace of one of the two tangent planes which can be drawn, and TV—T′V′ is the element of shade determined by it. In Pl. V., Fig. 18, as the ray, VN—V′E′ pierces the horizontal plane within the cone's base, on that plane, no lines can be drawn from N, tangent to the base. This indicates that no planes of rays can be drawn tangent to the cone. The lower nappe, therefore, would be wholly illuminated, but for the presence of the upper nappe, which casts a shadow upon it.

Fifth.—To construct the elements of shade on two intersecting cylinders, whose axes are at right angles to each other, in a plane parallel to the vertical plane of projection, one of these axes being vertical. Pl. V., Fig. 19. The elements of shade on the vertical cylinder are found at once, by drawing two vertical planes of rays, tangent to that cylinder. A*f* is the horizontal trace of such a plane, coinciding with the horizontal projection of a ray, and giving the element of shade A—A′A″.

The elements of shade on the horizontal cylinder, are found by revolving the plane of either of its bases either into, or parallel to, one of the planes of projection. Let the plane *u*D*v*′ be revolved about its vertical trace D*v*′, into the vertical plane of projection. The centre *n*,*n*′ of the base contained in this plane, will then revolve in the arc *nn*‴—*n*′*n*″, to *n*″. The circle, with *n*″ as a centre, and radius *n*″T″=*n*′E′, will then be the revolved position of the base *uy*—*e*′*v*′, and the projection of the entire horizontal cylinder upon the plane *u*D*v*′. Now, T″*k*″ is the trace on the plane of the revolved base, of a plane of rays perpendicular to that plane and tangent to the horizontal cylinder, T″*k*″ being drawn in any assumed direction, if the principal projections of a ray are not given. T″, and the point V″ diametrically opposite, are then the auxiliary projections of the elements of shade of the horizontal cylinder. By a counter revolution, in which these points, considered as the extremities of these elements of shade, revolve in the arcs T″*e*′—, *t*″*e* and V″*v*′—*v*″*p*, we find *e*H—*e*′H′ for the lower front element of shade, and *p*V—*v*′V′ for the upper back element of shade.

Remark.—If the principal projections of the light were given, $T''k''$, instead of being assumed, would be constructed as the auxiliary projection of the light upon the plane uDv', by an obvious construction.

§ II.—Shadows.

39. As in the case of planes, so in that of single curved surfaces, when all their elements are parallel, the projection of the surface will be a line, when it is perpendicular to a plane of projection. The projection of the perpendicular *plane* will, however, be a straight line, while that of the single curved surface will be a curve; viz. its curve of right section. As cylinders alone, among single curved surfaces, have parallel elements, they only can have all their elements perpendicular to the same plane, and hence they only can be projected in a line.

40. For all cases of shadows upon cylinders whose axes are perpendicular to a plane of projection, we have, from the preceding article, the following *special method. Determine, by inspection, the intersection*, M, *of a ray through any point*, P, *of the line casting the shadow, with the linear projection of the cylinder. This will be one projection of the shadow of* P. *Its other projection will be the intersection of the line perpendicular to the ground line, through* M, *with the other projection of the ray through* P.

The two following problems illustrate this method.

Problem XII.

To construct the shadow of the projecting head of a cylinder upon the cylinder.

Pl. IV., Fig. 16. The projections of this body have been fully described in the first case of the preceding problem. The right-hand edge, $G''A'$, of the head, above the element of shade, G—$G'G''$, casts the shadow required. Drawing the projection, Cd, of a ray, on the plane containing the linear projection, bdE, of the cylinder, d is at once found, as one projection of the shadow of C,C′ upon the cylinder. Its other projection, d', is the intersection of the perpendicular, d—d', to the trace (ground line) Fb, with the other projection, $C'd'$, of the ray through C,C′. The

other points of shadow are found in exactly the same way, as is evident by inspection. The tangent ray, D*e*—D′*e*′, determines that point of shadow, *e*,*e*′, in which the shadow is lost in the element of shade, *e*—*e*′*e*″.

Problem XIII.

To construct the shadows of the edges of shade of a cross upon a cylinder, both bodies being seen obliquely.

Principles.—Pl. IV., Fig. 17. The three projections of the cross and cylinder, of the ray of light, and of the elements of shade of the cylinder, having been already constructed, as explained in Prob. XI. (*Third*), we may proceed at once with the construction of the shadow, after having, *first*, determined, by inspection, the edges of shade of the cross, and, *second*, the limits of that part of the body of the cross, whose edges of shade cast shadows on the cylinder.

The edges of shade must, evidently, here be determined with reference to a single direction of the luminous ray in space; that is, we must find the projections of those edges, on the three planes of projection, which in space really cast shadows, when the light comes in the single direction projected in FL—F′L′—G″L″. These edges are, in the present case, B″D″—BD—B′D′; the one diagonally opposite, through YY″; then *c*″—*c*‴P—*c*′P′, and the one diagonally opposite, G″—FG—F′G′; also X″G″—XG—X′G′; its diagonally opposite edge, *c*″*w*″—*c*‴*w*; next X″*c*″—XP—X′P′; and its diagonally opposite edge, F*w*—G″*i*, and the horizontal edge at D″,D,D′.

The limits of that portion of the cross, which casts a shadow on the cylinder, are found by drawing the two tangent planes of rays, whose traces are M″J″ and T″H″, and which determine the elements of shade on the cylinder (Prob. XI. *Third*). These planes contain those points of the cross, which cast shadows on the elements of shade, where the shadow disappears from the cylinder.

Construction.—The point M″,M,M′, cut from the cross by the plane M″J″, casts a shadow on the element of shade, at a point whose auxiliary projection is J″, whose horizontal projection may be found by projecting J″ into the horizontal projection, M*a*, of the ray through M″,M, at *a*; and whose vertical pro-

jection may then be found by projecting a into the vertical projection, $M'a'$, of the same ray, at a'. Otherwise: we may construct the vertical projection, $J'K'$, of the element of shade, on which the shadow of M'',M' is known to fall, as explained in (Prob. XI. *Third*). Then the intersection, a', of this element with the ray, vertically projected in $M'a'$, is the vertical projection of the required point of shadow, which may be then projected into the horizontal projection, Ma, of the same ray, at a.

Thus every point of shadow may be found, first in horizontal projection, and then projected up; or first in vertical projection, and then projected down. Also, a may be projected up into $J'K'$, instead of into $M'a'$; or, a' being first found, as above, it may be projected down into JK, to find a. Indeed this is, practically, the most exact construction.

As all the other points of the shadow are found precisely as just described, we shall only point out those, which, in the present figure, it is desirable to find. This will assist in solving the problem, when the relative position of the bodies, the light, and the planes of projection, are slightly changed.

The shadow on the foremost element, b''—vN—$v'N'$, is cast by the point, R'', R,R', in the ray $b''R''$, and falls at b'',b,b'.

In the figure, the cross is made tangent to the cylinder along the edge c''—$c'''P$, which is therefore its own shadow. The highest point, S'', of the edge $B''X''$, which casts a shadow, is found by drawing the ray $c''S''$. Its shadow is c'',c,c'. The fragment, $S''X''$, casts the shadow, $c'l'$, on the front of the short arm of the cross.

The shadow on the highest element, is found by drawing the ray $p''Q''$, which determines the point Q'',Q,Q', whose shadow is p'',e,e'. The edge $X''c''$—XP casts the shadow Pd—$P'd'$; the edge XG, the shadow from d to the point just beyond e; the edge GF, the shadow from the latter point to o; the edge through F and w, the shadow op, and the edge through c'' and w'', the shadow pc''', partly invisible. The portion $G''T''$—UT, of UD, casts the shadow gf, and the portion, $V''i$, of the lower left hand edge, the shadow kh.

Remarks.—*a.* The portion of shadow cast on the horizontal plane in front of the cylinder is shown. It is found as in Prob. III. The shadow of the cylinder, and of the part of the cross which is beyond the plane $H''T''$, can be found on the horizontal

plane in the same manner, and will add to the beauty of the figure, especially when shaded in graduated tints.

b. If the light were to make a greater angle with the horizontal plane than the edges parallel to X″c″, as seen in the auxiliary projection, the edges, as X″—X′l′, would be edges of shade.

c. Those edges are inked as lines of shade, in the auxiliary elevation, which are so, in reference to the single direction of the luminous ray, shown in this figure.

41. The *general method* for finding shadows on *oblique* cylinders, and other single curved surfaces, is this: *Pass planes of rays so that each shall cut the line casting the shadow in a point* P, *and the single curved surface in a straight element,* E—*or in some other simple section. The point,* S, *where the ray through* P *meets the element,* E, *is the shadow of* P *upon* E, *and is therefore one point of the shadow of the given line upon the given surface.*

Remarks.—*a.* For the cylinder; a plane of rays must be parallel to the axis, in order to intersect the cylinder in straight elements.

b. For the cone; a plane must contain the ray through the vertex, in order to intersect it in straight elements.

c. The two following problems illustrate this general method.

Problem XIV.

To construct the shadow cast by the upper base of a hollow oblique cylinder upon its interior.

Principles.—The method of (41) applied to this problem gives the following principles of solution. Any secant plane, parallel to the plane containing the axis and a ray, will cut two rectilinear elements from the cylinder, one of which will be towards the source of light. The upper extremity of this element casts a point of shadow on the opposite element in the same plane. The elements contained in the same tangent planes of rays, coincide in one, which is the element of shade. Hence the upper extremity of this element is also a point of shadow on the interior of the cylinder.

Constructions.—Pl. III., Fig. 13. According to the foregoing principles, *f,f′*, and *u,u′*; the upper extremities of the elements

of shade (Prob. XI. *Second*) are the points where the required shadow on the interior begins.

To find any other point of this shadow. The plane of rays whose horizontal trace, MH, is parallel to the horizontal trace, NEL, of the plane of rays through the axis, contains the elements M*g* and H*e*. The point, *g*,*g'*, of the former element, casts its shadow on the latter element, at *h*,*h*. This point may be found by drawing the horizontal projection, *gh*, of a ray, noting the point *h*, and then projecting it, either into the vertical projection, *g'h'*, of the ray, or into the vertical projection, D'*e'*, of the element containing it. Otherwise: the same point may be found by first drawing the vertical projection, *g'h'*, of the ray through *g*,*g'*, noting its intersection, *h'*, with the vertical projection, D'*e'*, of the element containing it, and then projecting it either into *gh*, the horizontal projection of the same ray, or into H*e*, the horizontal projection of the element containing *h*,*h'*.

To find the shadow *cast by any particular point*, assumed in advance, on the upper base, as *n*,*n'*. Draw the element, *n*—N, containing this point, and through N, its intersection with the horizontal plane, draw the trace, NE, of a plane of rays, and then proceed as before, to find *o*,*o'*, the shadow of *n*,*n'*.

Conversely, to find the shadow *cast upon any element*, assumed in advance, as C*b*—C'*b'*. Through the intersection, C, of this element, with the horizontal plane, draw the trace CR, parallel to NE, of the plane of rays which cuts from the cylinder the element R*r*, towards the source of light; whose upper extremity, *r*, casts the shadow, *s*,*s'*, on the assumed element *b*C—*b'*C'.

The lowest point, *o*,*o'*, of the shade, is evidently on the element, E*d*—E'*d'*, contained in the plane of rays through the axis, because the two elements contained in this plane are the farthest apart, being diametrically opposite.

Remarks.—*a*. It does not strictly belong to this problem to construct the shadow of the cylinder on the horizontal plane. It is however shown in the figure, and is bounded by BB'', the shadow of the element of shade, B*u*—B'*u'*; the semi-circle, B''F'', whose centre is L, and which is the shadow of *udf*, and FF'', the shadow of the element of shade, F*f*—F*f'*.

b. The vertical projection of the shadow on the interior of the cylinder is invisible.

c. It happens in the present figure, that the horizontal projection, *fms*, of the same shadow is a straight line. This shows

that this shadow is a plane curve, whose plane happens, in this instance, to be vertical. The shadow being a plane curve, and also the intersection of two cylinders having a common base, *f*JI—J′I′,—viz. the given cylinder, and the cylinder of rays through its upper base—we infer, what is actually true, that the intersections of such cylinders, having, as they do, *two common tangent planes*, are plane curves.

d. The last remark serves as a simple illustration of the use of the exact constructions of Descriptive Geometry as a means of research in discovering theorems, particularly of form and position, relating to geometrical magnitudes.

42. In the following problem, besides an illustration of the *general method* of (41) there is an illustration of the *special method* called *the method of one auxiliary shadow*, which is as follows:

Where the shadow of a line on any surface meets the intersection of that surface with a second surface, is a point of the shadow of the given line upon that second surface.

Thus: where the shadow of a staff, upon the ground, meets the intersection of the ground with a house, is where the shadow of the staff upon the house begins.

Problem XV.

To construct the shadow of the upper base of a vertical right cone, upon the lower nappe of the same cone.

Principles.—According to (41) the upper base is the line casting the shadow, the lower nappe is the single curved surface receiving the shadow, and any secant plane containing the ray through the vertex, and cutting the base, is a secant plane of rays containing two elements of the cone. The intersection of each of these elements with the upper base, is, in general, a point casting a shadow on the other element.

Construction.—Pl. V., Fig. 18. AGL and A′H′—V′—A″H″ are the projections of the cone, and VN—V′E′, of the ray through the vertex, which pierces the horizontal plane at N,E′, and the plane of the upper base at D‴,D″. As all the planes of rays are to contain this ray, their traces on the plane of the lower base will all pass through N, and those on the upper base, through D‴, as seen in horizontal projection.

To find the points of shadow, which fall on the circumference of the lower base of the cone. These points are found by the *special method* of (42). The ray VJ—V″J′, through the centre of the upper base, pierces the horizontal plane (the plane of the lower base) at J′. Then a circle with J as a centre, and radius JL equal to V″H″, that of the upper base, will be the shadow of the upper base on the plane of the lower base. LG is an arc of this shadow, and L,L′ and G,G′, its intersections with the lower base, are the required points of shadow on the circumference of that base. Other points might be similarly found.

Remark.—It will be observed that, by this method, we necessarily find, neither the elements nor the rays, containing these points of shadow; nor, consequently, the points casting them. To find the *elements*, connect L,L′ and G,G′ with the vertex V,V′. To find the *rays*, draw the projections of rays through these points and note their intersections with the corresponding projections of the upper base. The latter intersections will be those *points* of the upper base, whose shadows are L,L′ and G,G′.

To find other points of the required shadow. These are found by the general method of (41). Let ANI be the horizontal trace of any secant plane of rays. It cuts from the cone the two elements IVB, and AH—A′V′H″. The point B,B″ of the former, in the upper base, casts a shadow on the latter at *n′*,*n*, found by drawing the vertical projection B″*n′* of a ray, and projecting its intersection, *n′*, with A′V′, at *n*, on AV, which last point may also be found as below.

Again: let AF be the horizontal projection of the trace of a secant plane of rays, upon the plane of the upper base. This plane cuts the upper base at A,A″, and the cone in the element FVM—M′V′ receiving the shadow of A,A″, at *a*,*a′*. This point was found by drawing the horizontal projection, A*a*, of a ray, and projecting *a* into M′V′ at *a′*. The point, *a*, might, instead, have been projected into the vertical projection, A″*a′*, of the same ray. It might also have been found as *n*,*n′* was. (See *a*,*a′* in Pl. IV., Fig. 17.)

When the secant plane of rays, as at EK, is perpendicular to the vertical plane of projection, the vertical projections of the elements, and rays, contained in it, are confounded together in its vertical trace E′D″. Hence we have not, in this case, the choice of methods of procedure, given above, but *must* find, first, the horizontal projection of the point of shadow contained

in this plane. Thus: the plane ENK cuts from the cone the elements EV and KVD; the latter of which determines the point D,D″ on the upper base, which casts a shadow on the former at b,b'; found only by drawing the ray Db, and projecting b into the vertical projection, E′V′, of the element EV, at b'.

The highest point of the shadow, is in the vertical meridian plane of rays, CVJ. The point C,C″, cut from the upper base by this plane, casts a shadow on the element CV – C′V′, in the same plane, at e,e'.

Through the points now found, the projections of the curve of shadow can be sketched. In the horizontal projection, the upper nappe conceals the shadow. In the vertical projection, the whole of the front of the upper nappe, and all of the lower nappe, above the line of shadow $n'b'$G, are shaded, being visible portions of the darkened surface of the cone.

Remark.—If the light had made a less angle with the horizontal plane, than that made by the elements of the cone, there would have been elements of shade on the cone and a shadow cast by the upper circle on the interior surface. It is recommended that the problem should be solved under these conditions.

43. In the following problem occurs an illustration of the *special method* of (40), of the *special method* of (42), and of a new *special method*, which may be called *the method by one indefinite rectilinear projection of the shadow, known by inspection.* Thus the shadow of a vertical straight line, upon any surface, will be straight, as seen in horizontal projection. The method itself, as applied in finding any other projection of the shadow, is this. *Having the intersection of the given projection of the shadow with any element of the surface on which it falls, project that point upon the other projections, one or more of the same element.*

Problem XVI.

To construct the shadow cast by the vertical cylinder (Prob. XI., Fifth) upon the horizontal cylinder, and by the horizontal cylinder upon the vertical cylinder.

Principles.—The lines of each cylinder, which cast the shadows, are their elements of shade, and the portions of the circumferences of their bases, which are curves of shade, as found

in (Prob. XI., Fifth). The shadow of the horizontal cylinder upon the vertical one, will be found by the *special method* of (40). The shadow of the upper base of the vertical cylinder on the horizontal cylinder will be found by the *special method* of (42), and the shadow of the element of shade of the vertical cylinder, upon the horizontal cylinder, will be found by the *special method* of (43).

Constructions.—1°. Pl. V. Fig. 17. *To find the shadow of the lower front element of shade,* eH—e′H, *of the horizontal cylinder, upon the vertical cylinder.* This shadow begins at t,t', where this element of shade pierces the latter cylinder. Through any point, X,X′, of the element of shade, pass a ray $Xx-X'x'$. This ray meets the vertical cylinder in a point, whose horizontal projection is x, and whose vertical projection is x', the intersection of the perpendicular, $x—x'$, to the ground line, with the vertical projection, $X'x'$, of the ray. In the same manner, s,s', the shadow of the extremity of the element of shade, is found; also, r,r', the last visible point of shadow on the vertical cylinder, which is cast by the point, q,q', of the left hand base, found by drawing the ray, $rq-r'q'$, backwards from r, the horizontal projection of the extreme left hand element of the vertical cylinder. Through the points now found, $t'x's'$, the line of shadow, cast by $et-e't'$, may be sketched; also, $s'r'$, the shadow of the portion, $eq—e'q'$, of the base, just mentioned.

2°. *To find the shadow of the upper base of the vertical cylinder, upon the horizontal cylinder.* Assume any horizontal plane, as the one whose vertical trace is V′v', and containing elements of the horizontal cylinder, in which it is supposed that points of the required shadow will fall. This plane cuts two elements from the latter cylinder, the hindmost of which is projected at V′v'—V″. By making a counter-revolution, V″v'—$v''p$, of V″, to v',p, we find pV, the horizontal projection of this element. O′D′—OD is the ray through the centre of the upper base, and it pierces the plane V′v' at D′,D. A circle described with D as a centre, and radius equal to O′I′, is the shadow of the upper base on the auxiliary plane V′v'; then d, the intersection of this shadow with the element, pV, cut from the cylinder by that plane, is the horizontal projection of a point of the shadow of the upper base of the vertical cylinder upon the horizontal cylinder. By projecting d, at d', into the vertical projection, V′v', of the element in which it lies, we have both projections of this point

of shadow. The point b,b' is found in the same manner on the element mb—$B'b'$ contained in the auxiliary horizontal plane whose vertical trace is B'—b'. In like manner the point c,c', on the highest element, nF—E′F′, is found; the shadow of the centre of the base O′I′, on the horizontal plane containing the highest element, being at C′,C.

3°. *To find the shadow of the front element of shade*, A—A′A″, *of the vertical cylinder, upon the horizontal cylinder.* By (43) we have, at once, the straight line AK, coinciding with the horizontal trace of a vertical plane of rays through A—A′A″, for the horizontal projection of this shadow. It begins at A,h', where the element of shade, A—A′A″, pierces the upper half of the horizontal cylinder. Any other point, as g', of the vertical projection, is thus found. Assume any element, as g'J′, whose projection on the end elevation is g''. By a counter-revolution, the element through g'' is found in horizontal projection at Jg. Project g, its intersection with the shadow, AK, into the vertical projection, J′g', of the assumed element, and g' will be determined as required. Likewise a,a', on the highest element, and f,f', on the back element of shade, produced, are found. The curve $h'a'f'$, which, being an oblique plane section of a cylinder, is an ellipse, can next be sketched through the points now found. To find how far this shadow is real, draw the vertical projection, A″o', of the ray through the highest point A,A″, of the element of shade A—A′A″, note its intersection, o', with the indefinite shadow $h'a'f'$, and project o' at o. Then Ao—$h'g'o'$ is the definite shadow of A—h'A″ upon the horizontal cylinder. At o,o' begins the shadow, $ocbd$—$o'c'b'd'$, of the upper base, which is lost in the shade of the cylinder at d,d'.

Remarks.—*a.* When, as in this case, the front element of shade of the vertical cylinder lies to the left of the point, L′, a minute shadow will be cast by it on the lower half of the horizontal cylinder near L′. When, however, the same element of shade lies to the right of L′, a small portion of T″—L′H′ will cast a small shadow on the vertical cylinder.

b. By constructing the base G′I′ in end elevation, which would be a straight line equal to G′I′, and in G′I′ produced, and by constructing, in the same elevation, the element of shade A—A′A″, both of the shadows on the horizontal cylinder could have been found by the simple special method of (40), applied very nearly as in Prob. XIII.

PROBLEM XVII.

To find the axes of an elliptical shadow, two of whose conjugate diameters are known.

Principles.—Pl. VI., Fig. 66. It has already been shown (Elemen. Proj. Drawing, Div. IV., Chap. V.) and (Descrip. Geom.) that, if the projecting lines of an object are oblique to a plane of projection, surfaces parallel to that plane will be shown, still, in both their true form and size; and that lines perpendicular to that plane will be shown as parallel lines, whose direction will depend on that of the oblique projecting lines.

In case the projecting lines make an angle of 45° with the plane of projection, as in "Cavalier Perspective" or Cabinet Projection, the projection of a perpendicular to the plane of projection will be equal to the line itself; since the line, its projection, and its projecting line, taken together, will form an isosceles right-angled triangle. But if the projecting lines make any other angle than 45° with the plane of projection, the projection of the line will be longer or shorter than the line itself. The latter case is shown in Fig. 66, which is a general oblique projection of a cube; FG, etc., being less than EF, the edge parallel to the plane of projection.

To view this figure, now, with reference to the present problem, the ellipse *mQnp* is the oblique projection of the circle inscribed in that face of the cube which is perpendicular to its front face, EFLR, in the line LR. Hence the figure also truly represents, in a pictorial manner, the projection of the circle MPNQ, considered as vertical, upon the horizontal plane LRHK; and, moreover, if O*o*, which thus represents an oblique projecting line, is regarded as a ray of light, then *mQnp* will represent the shadow of MPNQ on the plane LRHK.

Construction.—Let *mn* and *p*Q, Fig. 67, be given conjugate diameters of an elliptical shadow, represented by oblique projection, as diameters of the shadow of a vertical circle, upon the horizontal plane which is represented by the part of the paper below A′D′; a parallel to *mn*, through Q, as in Fig. 66. Then we have *om*, equal and parallel to the radius of the original circle casting the shadow, and Q, as the shadow of the foot of the vertical diameter of the same circle. Then erect QP, perpendicular

to A′D′, and equal to *mn*, and $OQ=\frac{1}{2}PQ$ is the radius of the original circle.

Next, lines that are parallel in reality, as ON and a tangent at P, are parallel in projection; and conjugate diameters are, each, parallel to the tangents at the extremities of the other. Hence *any* diameters at right angles to each other, in MPNQ, will cast shadows which will be conjugate diameters of the given ellipse. We therefore seek a pair of such diameters, whose shadows shall be conjugate diameters at right angles to each other, for these will be the axes required.

Now, since the shadows of the indefinite radii of MPNQ begin in A′D′, these radii, and their rectangular shadows, must be inscribed in two semicircles, having a common diameter in A′D′. Hence O*o* is a chord of the circle A′OD′*o*, thus composed, and *g*G, perpendicular to it at its middle point, meets A′D′ in the centre, G, of this circle. Then OA′ and OD′ are the rectangular radii, whose shadows, A′*o* and D′*o*, are at right angles to each other, and are each parallel to the tangents at the extremities of the other. Hence, limiting them by the rays *a*A and *d*D, and making *o*B=*o*A, and *o*C=*o*D, we have AB and CD for the required axes of the elliptical shadow *m*Q*np*.

Remark.—The construction just given is often found among plane problems on the conic sections, as it can readily be explained by the principles of plane geometry. But, as required by the spirit of the present subject, it is here explained by the principles of *projections* and of *shadows*.

44. The problem of the Niche is a favorite one in Shades and Shadows, owing to the considerable number of points of peculiar interest connected with it. The niche is a familiar concavity in the walls of halls, staircases, etc., and consists usually of a vertical half cylinder of revolution, united by its upper base with a concave spherical quadrant. The problem of the niche, as a problem of shadows, consists in finding the shadow of the edge of shade of the cylindrical part upon that part, and upon the lower base; and the shadow of the front semicircle of the spherical part, upon that part, and upon the cylindrical part.

45. Agreeably to the classification of (12, 13) it becomes desirable to divide this problem, and to retain here the construction of the shadows on the cylindrical part, only leaving the shadow on the spherical part to be placed with shadows on other double-curv-

ed surfaces. One of the topics of special interest, in this problem, is the direct construction of that point of shadow which falls on the curve of the upper base; and there are several such constructions, but which are applicable only when the spherical part is included in the problem. Hence it becomes quite desirable to find also a direct construction of the point in question which shall be applicable in the problem as given below, where the spherical part is not recognised. Such constructions will be found in the following problem.

Problem XVIII.

Having a vertical semi-cylinder with its meridian plane parallel to the vertical plane of projection, it is required to find the shadow cast on its base and visible interior, by its edge of shade, and by a vertical semicircle, described on the diameter of its upper base.

Principles.—All the points of shadow, save that on the circumference of the upper base (45), are found by the special method of (40).

Constructions.—Pl. VI., Fig. 20. A*d*B—A′A″B′B″ is the vertical half cylinder; AB—A″F′B″ is the vertical semicircle, whose shadow is to be found, and A*a*—A″*a*″ is a ray of light.

1°. *To find the shadows of the vertical line*, A—A′A″. A*a* is the horizontal trace of a plane of rays through the vertical line, A—A′A″. This plane will, therefore, cut the cylinder in an element whose horizontal projection is *a*. Hence, by drawing *a*′—*a*″, the vertical projection of this line, and the rays, *a*′*b*′ and A″*a*″, we find the whole rectilinear shadow, *a*′—*a*″, on the cylindrical surface, also the portion A″*b*′ of the line A′A″, which casts the shadow *a*′—*a*″. The shadow of A′*b*′, on the base of the cylinder, is A*a*.

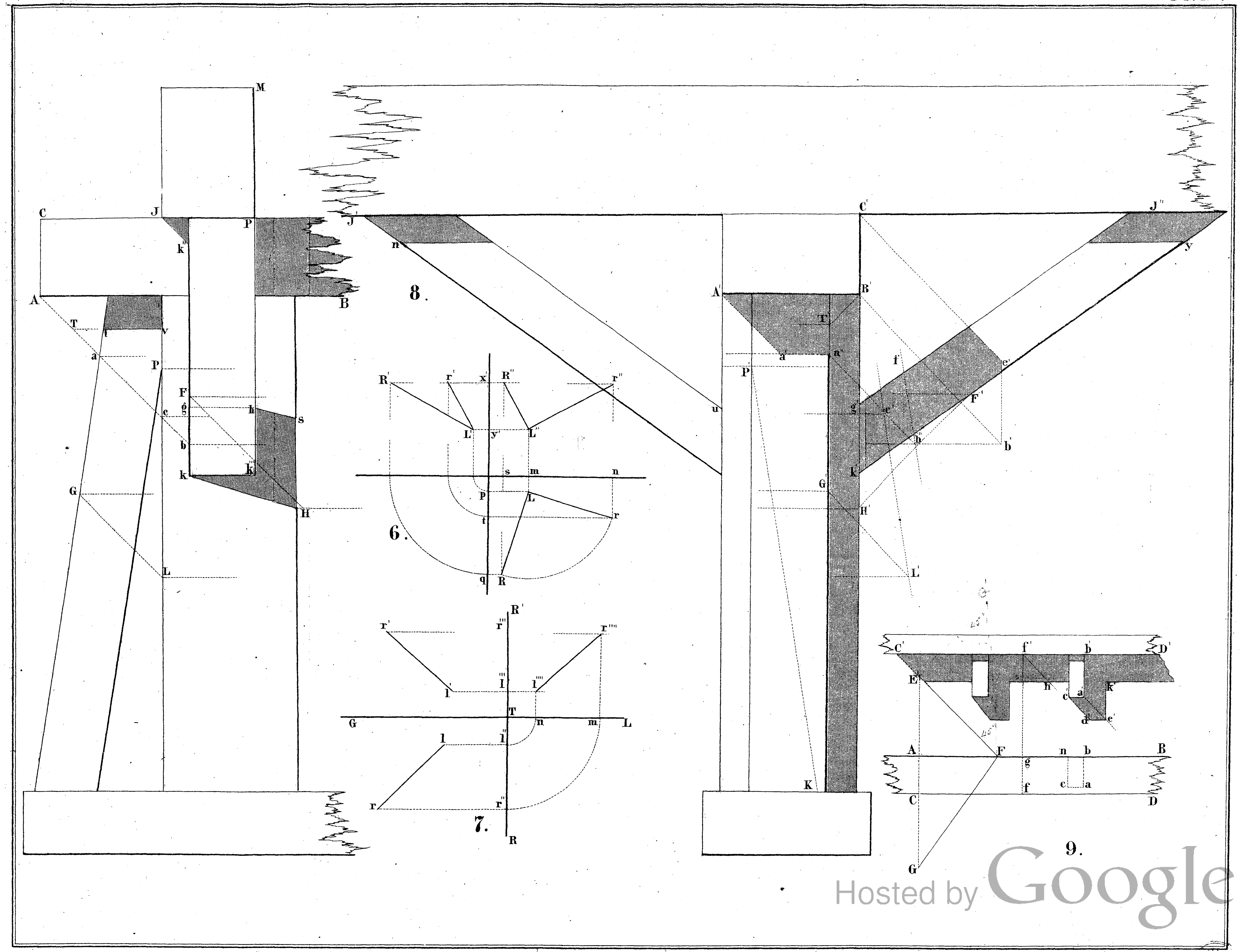
2°. *To find points of shadow cast by the semicircle* AB—A″F′B″. C*c*—C′*c*′ is a ray, through any assumed point, CC′, of this semicircle. It pierces the cylinder at the point whose horizontal projection is *c*, and vertical projection *c*′ (40); *cc*′ is therefore the required shadow of CC′. Other points of this shadow are found in the same manner.

3°. *To find the point of shadow on the upper base of the cylinder*, we will first use the following *special method of auxiliary oblique projections.* The ray CS—C′S′ pierces the plane of the upper base of the

cylinder at S′S. If then the semicircle, AB—A″F′B″, be revolved forward about AB—A″B″ as an axis, till it coincides with the plane of the upper base, CC′ will appear at C″, and C″S will be an oblique projection of the ray CS—C′S′, when the projecting lines come directly forward and downward at an angle of 45° with the horizontal plane. Observing the limitation of C″S by CS, it may be described as being inscribed in a circle, U—SKC″, whose centre, U, is found at the intersection of AB with uU, the bisecting perpendicular of C″S. Now a parallel ray, similarly inscribed in the circle E—BAG″, will evidently be the similar representation of a ray which contains a point of the semicircle AG″B casting the shadow, and of the semicircle AdB receiving the shadow. The point on AdB will be the point of shadow sought. This ray, parallel to C″S, will be a homologous side of a triangle similar to SCC″, and similarly situated. From these things, it follows that the two rays will be to each other as the radii, UK and EA, of the circumscribing circles SKC″ and BAG″. Constructing this proportion at *qmo*, we find *mq* for the length of the parallel ray, *d*G″. Its extremity, *d*, is located by drawing E*d* parallel to US. Projecting *d* at *d′*, gives *dd′* as the required point of shadow on the circumference of the upper base of the cylinder.

Remark.—To give completeness to the figure, draw *d*G″ parallel to SC″. It will be equal to *qm*. Then make the counter-revolution of AG″B to its primitive position, when G″ will be found in its primitive position at GG′. Then drawing the projections, G*d* and G′*d′*, of the ray which determines *dd′*, we shall have the point GG′, whose shadow is *dd′*, and, in plan, the complete triangle G″G*d*, similar to C″CS.

4°. By another *special method*, that of *auxiliary spheres*, the ray CS′—C′S′ pierces the plane of the upper base of the cylinder at S′S, as before. A plane, perpendicular to this ray, at its middle point *rr′*, will intersect AB—A″B″ in the centre of a sphere of which the ray is a chord. Knowing (Des. Geom. 53*b*) that the vertical trace of this plane will be perpendicular to C′S′, the line *r′*T′—*r*T, where *r′*T′ is perpendicular to C′S′, and *r*T parallel to the ground line—is a line of the plane, hence T′T is a point of its trace on the plane of the upper base. This trace must be perpendicular to CS, hence it is the line TU. UU′ is then the centre of a sphere, whose radius is US, and in which the ray CS—C′S′ is inscribed.



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To find the ray similarly inscribed in a sphere whose centre is E,E′ and radius EA.

In the proportion UK : EA :: CC″ : GG″ (=gG′) we find, by a construction like *qmo*, the fourth term, and locate it as at gG′. Then draw the ray Gd—G′d′, which intersects AdB—A″B″ at dd′ the point sought.

Remarks.—a. By finding the shadow on the cylindrical surface produced, as at ff', the intersection of the sketched line of shadow, $a''c'f'$, with A″B″, gives d' approximately and indirectly. Such constructions are less satisfactory than determinate intersections, found by direct constructions, such as the two preceding.

b. The straight shadow, $a'-a''$, and the curved one, $d'c'a''$, are tangent to each other at a''; for $a-a'a''$ is the element of contact of a vertical tangent plane at a, and the surface of the given vertical cylinder, and Aa—A″a'' is the element of tangency of the vertical plane of rays, Aa, with the semi-cylinder of rays having the semicircle, AB—A″F′B″, for its base. Hence $a-a'a''$ is the intersection of the tangent planes to the two cylinders, and is therefore also the tangent line to their curve of intersection; viz. to the shadow $a''c'd'f'$. (D. G. 170.)

SECTION II.

SHADES AND SHADOWS ON WARPED SURFACES.

§ I.—Shades.

46. The line of shade on a warped surface will, in general, be a curve of some kind. Its points must, therefore, be separately found. But any point of a curve of shade may be regarded, either as the point of contact of a *tangent plane of rays*, or as that of a single *tangent ray*. But, again, since we naturally associate tangencies of surfaces to surfaces, and of lines to lines, a tangent line to a given surface at any point, is generally constructed as a tangent to some curve, lying on that surface and passing through the point.

47. Here it is to be remembered that, as the consecutive elements of a warped surface are not in the same plane, a plane, K, passed through any one of them, E, will, in general, intersect all

the others on each side of it, forming a curve, C, whose intersection with the element, E, is, as explained in Descr. Geom. (212, 342), the point of contact of the plane K, which is thus a tangent plane as well as a secant plane.

48. Hence we have two *direct general methods* (22) for determining any point of the curve of shade on a warped surface. One of these methods will be given here, the other presently.

First Method.—Construct the point of tangency of any plane of rays, tangent to the warped surface, by passing a plane of rays through any element, and noting the point of intersection, P, *of that element, with the curve of intersection of the tangent plane with the warped surface. The point of intersection,* P, *will be the point of tangency of the plane of rays, and hence a point of the required curve of shade.*

49. When the warped surface is of the second order, either a warped hyperboloid, or a hyperbolic paraboloid, the plane of rays passed through any element, intersects the surface in a second element, of the other generation (Des. Geom.), whose intersection with the given element is the point of tangency of the given plane.

In illustration of the method of (48) take the following simple case.

Problem XIX.

To construct the curve of shade on a hyperbolic paraboloid.

Principles.—By taking the planes of projection as plane directors, and one of the directrices vertical, the construction will be simple. The horizontal traces of the planes of rays containing the horizontal elements will be parallel to those elements, respectively; and will cut the horizontal trace of the surface in points of the elements which are parallel to the vertical plane.

Construction.—Pl. VI., Fig. 21. For the sake of definiteness, call the horizontal plane the plane director of the first generation, and the vertical plane, the plane director of the second generation. Let the vertical line A—A″A′ and the line A″G—A′G, in the vertical plane, be the directrices of the first generation. Then divide these directrices proportionally, according to the properties of the surface, and, for convenience, let these parts be equal

on each line. Then AA″—A′; AC—c″C′; AD—d″D′, etc., will be elements of the first generation. In order that the curve of shade shall be on the visible face of the surface, or real, in case of an opaque solid, formed and placed as in the figure, the light should come as shown at A7 in horizontal projection. Without constructing the intersection of the ray through AA′ with the horizontal plane, by means of a given vertical projection, let 7 be assumed as this intersection. The line 7*n*, parallel to AA″, will then be the horizontal trace of the plane of rays through AA″. Then *na*, drawn from *n*, the intersection of the trace, 7*n*, with the trace, AG, of the paraboloid, and parallel to the ground line, represents the element of the second generation contained in the plane A—7*n*. Its intersection, *a*, with the element AA″, is the point of contact of the plane of rays, and hence a point in the required curve of shade.

Since the vertical directrix at A was equally divided by the horizontal elements, the rays through the points of division, being parallel lines, meet the horizontal plane in points, on A7, which would divide that line into equal parts, and which are also points in traces of planes of rays through those elements respectively. Hence 6*p*, parallel to AB, and 1*e*, parallel to AD, are, for example, horizontal traces of planes of rays containing those elements. These planes also contain, respectively, the elements, *bp* and *ed*, of the second generation, which intersect the former elements at *b* and *d*, two more points of the curve of shade.

Other points of shade are similarly found. The vertical projection of *b* is at *b*′; of *d*, at *d*′, etc. AN is the horizontal projection of *k*′N′, the first element below the horizontal plane. 1K is the horizontal projection, parallel to AG, of the trace of the plane of rays through AG on the horizontal plane *k* N′. Then by drawing k*g*, as in the previous constructions, we find *g*, the point of shade on AG.

Remarks.—a. From the properties of this surface, the curve of shade, that is, geometrically speaking, the curve of contact of a cylinder with a hyperbolic paraboloid, is a parabola, and hence is also a parabolic curve in both projections.

b. The problem, as here given, shows the shadow, as on an awning or porch roof in an angle of a building, and of the form of a simple hyperbolic paraboloid.

50. *Second Method* (48).—*Pass any secant plane of rays through*

the warped surface and construct its curve of intersection with the surface. Then draw a ray tangent to this curve of intersection. The point of contact, thus found, will be the point of contact of the ray with the surface, and hence a point of its required curve of shade.

Either of the above *direct* and *general methods* (48, 50) can be easily applied to any of the general warped surfaces, with or without plane directors, and given by their elements, in constructions which are too simple to need formal illustration.

51. When the warped surface is a hyperboloid, particularly when it is a hyperboloid of revolution, the *indirect special method of auxiliary tangent surfaces* is applicable. This method is *indirect*, as involving an intermediate surface whose line of shade is more readily found than that of the given surface (22), and is *special*, as being applicable to certain particular surfaces (22). It may be stated as follows:

Make any circular section, C, *of the given warped surface of revolution, the circle of contact of an auxiliary tangent cone, having the same axis as the given surface, and whose elements of shade*, E and E′, *are easily found.*

The point, T, at which the *element* of contact, E, of the auxiliary cone with a *plane of rays*, intersects the *circle* of contact, C, of the cone and the given *surface of revolution*, is the common point of contact of the plane, the cone, and the given surface. That is, it is the point of contact of the *plane of rays* and the *given surface*, and hence is a point of the curve of shade on the latter.

Hence, the essential principle of the above method, stated abstractly, is this: *Two surfaces*, M and N, *being tangent in a curve of contact*, K, *the intersection*, T, *of the line of shade of* N *with the curve of contact*, K, *is a point of shade, common both to* M *and* N, *and hence a point of the curve of shade of* M.

Remark.—The method just explained might be applied in constructing the curve of shade of a warped hyperboloid; but as it will be exhibited in connection with an analogous double curved surface (Prob. XXVII.) it is here omitted.

Problem XX.

To find the curve of shade on the common oblique helicoid, in the practical case of the threads of a triangular-threaded screw.

52. *Principles.*—Under this head are here arranged a few preliminary matters, which will prepare the way for attention to the immediate object of the problem.

1°. *Description of the screw.* Pl. VII., Fig. 22.—Let the circle, whose radius is AG″, be the horizontal projection of a vertical cylinder, called the *cylinder*, *core*, or *newel*, of the screw. Let $w'''l'''T'''$ be any isosceles triangle, whose base, $w'''T'''$, coincides with an element of the cylinder, and which lies in a meridian plane of the cylinder. Let this triangle have a compound motion; of rotation about the axis, A—A′A″, of the cylinder, and of translation parallel to that axis; and let each of these motions be uniform. With such motions, each point of the triangle, as w''', l'', or T''', will describe a helix (Des. Geom. 308); the side $l'''T'''$, not shown, but corresponding to $w'''l'''$ (dotted), will generate an upwardly converging zone of an oblique helicoid, and the side $w'''l'''$ (dotted) will generate a downwardly converging zone of a similar helicoid. The entire triangle will generate the volume called the *thread* of the screw. The helicoidal zone generated by $l'''T'''$ is called the upper surface of this thread, and that generated by $w'''l'''$ is its lower surface. The curve, $w'''G'a'r'$, etc., generated by w''', is called an *inner helix;* while the curve, $l''b''l''$, generated by l'', is called an *outer helix.*

All the inner helices are horizontally projected in the circle $G''am$, and all the outer ones in the circle sqx. The vertical projection of a helix is found by dividing its horizontal projection equally, and projecting the points of division upon the successive equidistant horizontal lines, which mark its uniform ascending progress.

The contours of the helicoidal surfaces are curves, apparently tangent, as seen in vertical projection, to the helices. That portion of the contour which bounds a single zone is, however, so slightly curved that it is sufficiently accurate to represent it by a straight line tangent to an outer and an inner helix, as at $l'''w'''$ (the full line) and $l'''T'''$. By thus drawing all the visible contours, the projections of the screw will be completed. The dotted

line, $l'''w'''$, whose horizontal projection is AS″, is an asymptote to the contour of the helicoid, being an element of the cone director (Des. Geom. 333–4th) whose axis is A—A′A″, so that any meridian plane cuts elements from it and from the helicoid, which are parallel.

The core of the screw is shown at V″″E‴ for a short distance above the horizontal plane k''C″, which cuts off the screw. The intersection of the screw with this plane, is a spiral of Archimedes (D. G. 339*b*), and is found by dividing S″S‴ into any number, as eight, of equal parts, and the semicircle, S‴G″N″″, into the same number of equal parts; then circles, with A as a centre, drawn through the points of S″S‴, will intersect the radii through the corresponding points of S″G″N″″ in points of the spiral required.

2°. *To construct any particular element.—First Method.*—Any line, as A*ae*, is the horizontal projection of an element (D. G. 339), using *a*, a moment, to mark a point on A*e*. Project *a* at *a*′, and *e* at *e*′, then *ae*—*a*′*e*′ will be the projections of the portion of the indefinite element A*e*, which lies on the zone bounded by the helices through the points S″,S″″ and S‴,w'''.

Second Method.—Produce the element through S″,S″″ and S‴,w''' (not shown in vertical projection) till it meets the axis, A—A′A″, at a point which we will call 2. Then the element through any point, as *e*,*e*′, will meet the axis as far above 2, as *e*′ is vertically above S″″. This follows from the uniformity of the two motions of the element (1°).

Remark.—As the construction of a single point of the curve of shade of the screw is somewhat lengthy, by any method, the principles of each of the constructions now to be explained, will be given in immediate connection with the construction itself.

53. *The method by assumed helices.* This title is given to the *special method*, employed for finding those points of the curve of shade, which are upon any given, or assumed, helical section of the helicoid.

Principles.—The *principles* of this method may be thus stated. *First.* The tangent line and the element, through any point of the same helix, make, each, a constant angle with the horizontal plane—which, for convenience, is supposed to be perpendicular to the axis of the screw. Hence, *second*, the tangent planes to

the helicoid at all points on the given helix, make a constant angle with the horizontal plane, they being determined by the lines just mentioned. Hence, *third*, the angle included between the element in any tangent plane, and the line of greatest declivity of that plane, is constant for all the planes which are tangent at points on the same helix. (The line of greatest declivity of any plane, is evidently perpendicular to its horizontal trace.) *Fourth.* Any plane tangent to the cone director of the helicoid (Des. Geom. 339. Discussion) is parallel to one of the tangent planes to the helicoid, on the homologous element, that is, on the element contained by a meridian plane passed through the element of tangency of the cone director.

The method, based on these principles, consists of the following operations. Draw any plane, P, tangent to the helicoid at a point of the given helix, and determine its line of greatest declivity, L. Then draw a *plane of rays* tangent to the cone director, and determine its line of greatest declivity, L′; which will be parallel to that of the tangent plane above mentioned (*Fourth*) to the helicoid. Then the element, whose angle with L′ equals the angle made by L with the element of contact of P, will be the element whose intersection with the helix will be the required point of contact of a plane of rays with the given helix, that is, the required point of shade.

Construction.—Pl. VII., Fig. 22. 1°. *Let the assumed helix be the outer helix*, S″*eq*—S‴*e*′. At any point, *o*,*o*′, of this helix, draw the tangent line, *o*O, and the element as A*o*. The tangent pierces the horizontal plane of projection at O, where *o*O is equal to the arc *os*S″ (Des. Geom. 317), and the element pierces the same plane at F,F′; hence FO is the horizontal trace of the tangent plane to the upper surface of the thread at *o*,*o*′. Now AG, perpendicular to FO, is the line of greatest declivity of this tangent plane, and GAF represents the angle between this line and the element of contact, AF, of the tangent plane.

Any cone whose axis is A—A′A″, and whose elements make an angle with the axis, equal to that made by the elements of the helicoid with the same axis, may be taken as the cone director of the helicoid. Then let AI—A″I′ be the generatrix of the cone director. The circle with A as a centre, and radius AI, will be the base of this cone. Through its vertex, AA″, draw the ray AB—A″B′, which pierces the horizontal plane of projection at B′,B, hence BD is the horizontal trace of a plane of rays, tangent

to the cone director along the element AD, which is also the line of greatest declivity of this plane, and is the horizontal projection of the parallel element on the helicoidal thread. Then lay off, on the outer helix, and from AD, an arc $1b$, equal to $u'''o$, and the point b will be found as the point of contact of a plane of *rays*, tangent to the upper surface of the thread, at a point on the outer helix. Hence b, and its vertical projection, b', are the projections of the required point of the curve of shade on the upper surface of the thread and on this helix.

BC is the horizontal trace of another tangent plane of rays to the cone director of the upper surface of the thread, and CA is its element of contact, and line of greatest declivity. Therefore, make $C''''q$ equal to $u'''o$, and q,q' will be another point of the curve of shade of the upper surface of the thread, and on the outer helix.

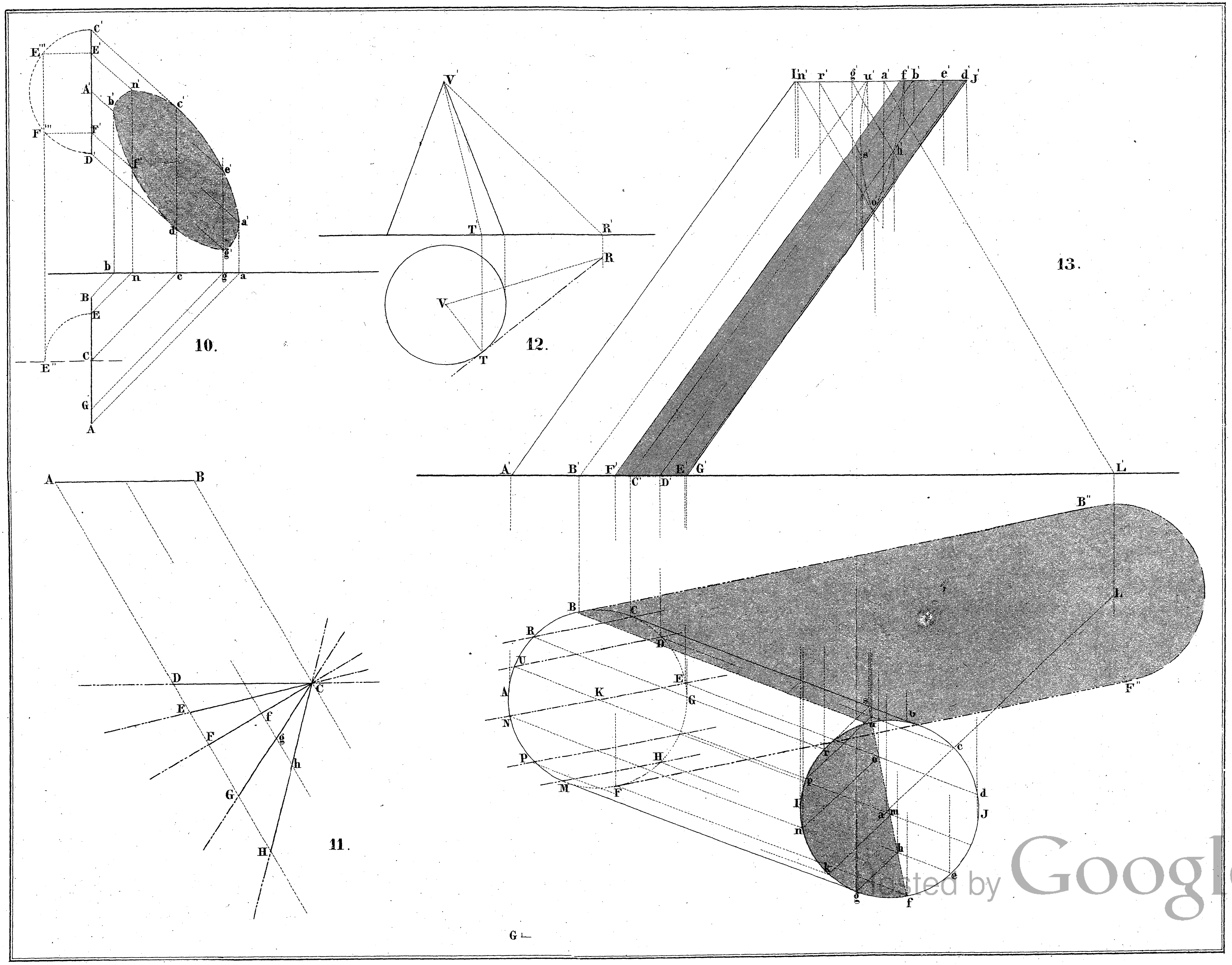
2°. *To find a point of the curve of shade on the upper surface of a thread and on the inner helix*, $S'''G''a—w'''G'a'$. To simplify the construction, take an auxiliary horizontal plane, $p'J'$, as far below $n,'n$, on the element Ao, before used, as the horizontal plane of projection is below o,o'. Then, drawing AO, the line nP will be the horizontal projection of a tangent at n,n', which pierces the plane $p'J'$ at P. For, by the properties of the helix and its tangent, $nP = nG''S'''$ and $oO = osS''$, while

$$nG''S''' : osS'' :: An : Ao$$
$$:: AP : AO$$
$$\text{hence } nP : oO :: AP : AO,$$

a proportion which is evidently constructed by limiting the parallel tangents, at n and o, by the straight line AO.

The element, An', pierces the plane $p'J'$ at p',p; hence pP is the trace, on the plane p'J, of the tangent plane to the thread at n,n'; and A2, perpendicular to pP, is the line of greatest declivity of this plane. Now, as before, BD is the horizontal trace of the plane of rays, tangent to the cone director of the helicoidal surface now considered, and AD is its line of declivity, and the horizontal projection of a parallel element on the helicoid. Then make $4a$ equal to $3n$ (53) and a,a' will be the required point of the curve of shade of the upper surface of a thread and on the inner helix. r,r' is a similar point of the other curve of shade, on the back part of the same zone.

Remarks.—a. Points on other helices, such as intermediate ones, can be similarly found.



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b. By going through with the preceding constructions, as applied to the lower surface of the thread, it will be found that, in passing from the element of contact to the line of declivity of a tangent plane, the motion will be, in a (rotary) direction, around A, opposite to the direction ou'''. Hence, DA, the line of declivity of the tangent plane of rays to the cone director, being produced to x, Ax is the horizontal projection of the element of shade of a cone director of the lower surface of a thread, and of the parallel element on the thread. This being so, we proceed, according to the former statement, to make xl equal to $1b$, and ym equal to $4a$; then lm is a curve of shade on the lower surface of a thread. Likewise, hj is the horizontal projection of a second curve of shade on the same surface.

c. Each of the curves of shade, now found in horizontal projection, has as many vertical projections, and corresponding curves in space, as there are helicoidal zones in the assumed portion of the screw. Thus, ab is vertically projected at $a'b'$, and $a''b''$; qr, at $q'r'$ and in part near g' and s'; lm, at $l'm'$ and $l''m''$; and hj, at $h'j'$ and in part at $k''j''$, and m'''.

d. If we conceive the entire helicoid, of which the upper surface, as $S''''w'''a'e'$, of any thread is a zone, to be represented, we may proceed as before to find other points of its curve of shade. The point A,D′, where the curve of shade meets the axis, is the intersection of the axis with the element through which the meridian plane of rays, RAK, is passed, since, as the axis and this element are the two lines cut from the surface by a plane of rays containing the supposed element, their intersection is the point of tangency of the plane of rays (D. G. 342), and is therefore a point of the curve of shade. We thus find $baAjh$—$b'a'D'$ (the rest of the vertical projection not shown) for a complete branch of the curve of shade on the helicoidal surface bounded by a helix of a radius equal to AS″. Likewise $lmArq$—$l'm'$..... is the branch of the curve of shade, on the similar helicoid of which the lower surface of a thread is a zone.

e. The point, A,D′, is also the point of tangency of the plane of rays RAK, from the particular properties of the helicoid, as well as from those of warped surfaces in general, as explained in the last remark. For the axis A—A′A″ is the limiting helix, generated by that point of an element which is in the axis; but, being straight, it is its own tangent, throughout its whole extent; hence, A,D′, like other points of contact, e,e', etc., of tangent

planes, is the intersection of a tangent to the helix, through A,D′, with the element through the same point.

54. A little consideration of the properties of the helicoid, explained in Descriptive Geometry (Des. Geom. 339), will enable the student to understand the following additional properties, which are here applied in another construction of the required curve of shade, by the *special method*, called the *method by assumed elements*.

Principles.—The development of a given portion of any helix —its axis being conceived to be vertical—is the hypothenuse of a right-angled triangle, whose altitude is the vertical height between the extremities of the given portion, and whose base is the development of the circular arc forming the horizontal projection of the same portion.

55. Portions of different helices on the same helicoid, but included between the *same meridian planes*, may be called homologous. Points in which they intersect the *same element*, may be called homologous, or corresponding points.

Since all points of the generatrix of a helicoid ascend at the same rate, the homologous portions of helices just mentioned, will be of equal height, between their extremities; that is, they will be included between pairs of equidistant horizontal planes. Tangents to these helices *at corresponding points*, and limited by the horizontal planes which include these helices, may be called homologous tangents. Their horizontal projections will be parallel, and equal to the concentric circular arcs which are the horizontal projections of the respective helical arcs to which they are tangent. But these arcs, and hence their tangents, are proportional to their radii; hence the horizontal projections of these tangents will terminate in a straight line through the horizontal projection of the axis (53–2°), which, with the tangents, and their respective radii, will form similar triangles.

56. Once more, equidistant horizontal planes will intercept equal portions of any element; hence, if two such planes contain, respectively, the point of contact and the trace of a tangent plane through any element; and if another pair of such planes contain the point of contact and the trace of another tangent plane through the same element, the distance between the trace and the point of contact of one tangent plane will be equal to the distance between the point of contact and the trace of the other

tangent plane, these distances being measured on the element contained in both of these planes. Recollecting, too, that the helix coincides with its tangent when developed, we have the following.

Construction.—Pl. VII., Fig. 22. Assume any element, A*e*, between the previously found points, *a* and *b*, on the helices. Making the tangent, *e*L, equal to the arc *es*S″; the point L is the intersection of this tangent with the horizontal plane of projection (D. G. 317). Then H′H being the intersection of the element A*e*—E′*e*′ (52–2°) with the horizontal plane, HL, is the horizontal trace of the tangent plane to the screw, at *ee*′ (D. G.347).

Next, by drawing a ray (not shown in vertical projection) through A,E′, it will pierce the horizontal plane of projection at 6 in the line AB, and 6HM will be the horizontal trace of a plane of rays through the assumed element.

Now, if the trace, HM, of this plane of rays, were found on a horizontal plane as far below its required point of contact, as the trace, HM, is now below *ee*′, this *new trace* would be parallel to HM, the *tangent* at its point of contact would be parallel to *e*M, and their distance apart, measured on A*e*, would be equal to *e*H. Hence the portion of the new tangent, cut off by the new trace, would be equal to *e*M. But all tangents to helices at points on A*e*, and included between pairs of equidistant horizontal planes, will terminate in AL, as seen in horizontal projection (53–2°); hence, transfer M to N, on AL, by a line MN parallel to HA, draw N*c*, parallel and equal to *e*M, and *c*,*c*′ will be the point of contact of the plane of rays, HM, with the screw, and on the assumed element A*e*. Hence *c*,*c*′ is a point of the required curve of shade.

Remark.—By the above method, as well as by that of (53–2°), we can find the indefinite curve of shade on a complete helicoid.

57. Those points of the curve of shade on the under surface of the thread, which are on its helical limits, might be found by the method of [53–2° (Rem. *b*.)]. For the sake, however, of illustrating another construction, they will be found by the following *special method*, which may be called the *method of helical translation of a tangent plane.*

Principles.—Construct a simple tangent plane, P, to the helicoid at any point of a given helix, find its trace, T, on any horizontal plane, K; and note its intersection, I, with the axis of the helicoid.

A ray through I will pierce the plane K in the trace, T, if P is a plane of rays. If this ray does not meet the trace T, revolve it about the axis of the helicoid, when it will generate a right cone, from which will be cut two elements, E and E′, by the plane P. By revolving either of these to coincide with the ray, and by revolving the point of contact of P through an equal angle, keeping it also on the helix, we shall have the point of contact of P, when translated, helically, so as to be a tangent plane of rays. This point of contact will therefore be a point of the required curve of shade on the given helix.

Construction.—Pl. VII., Fig. 22. Let the tangent plane be constructed at the point S″,l''', and let its trace be found on the horizontal plane S′K′. Thus S″A—l'''U, the element, through S″,l''', of the helicoidal zone, forming the lower surface of the thread considered, pierces the plane S′K′ at S′,S. The plane S′K′ being here taken at a distance above S″l''' corresponding to a quarter revolution of S″,l''' around the axis A—A′A″ we have S″T″″, equal to a quadrant of the circle S″eq, as the tangent at S″l''', piercing the plane S′K′ at T″″. Hence T″″ST is the trace, on the plane S′K′, of the tangent plane at S″,l'''. Now, as this tangent plane contains the element l'''U, it cuts the axis at U, and we wish next to find whether a ray through U pierces S′K′ in the trace TT″″. For lack of room on the plate, draw AK—UK′, the position of a ray through A,U, after a revolution of 180° from its primitive position, AR. (The angle K′UA′ is equal to the angle B′A″A′.) The revolved ray pierces the plane S′K′ at K′,K, one point of the base, KQT, of the cone generated by the revolved ray AK—U′K. As this base does not contain R, the ray in its primitive position, AR, does not pierce the plane S′K′, in the trace, TT″″, of the tangent plane. Hence this tangent plane is not a plane of rays. But AT and AQ are elements cut from the cone, just described, by the tangent plane, and may be considered as two positions of the ray AR, after being revolved about the axis, A—A′A″, till it becomes a line of the tangent plane. Therefore, the ray being at TA, when the point of contact of the tangent plane is at S″l''', make S″h=7—5, and project h at h', then will h,h' be the point of contact of the tangent plane after being revolved, with an ascending helical motion about A—A′A″, till by the return of TA to being a ray AR, it becomes a tangent plane of rays. Likewise, AQ being the revolved position of the ray, RA, when S″,l''' is the point

of contact of the tangent plane, make $S''l=8-5$, and project l at l', and l,l' will be another point of contact of the tangent plane, after moving, with a descending helical motion, till it becomes, again, a tangent plane of rays. Hence, finally, h,h' and l,l' are points of the two curves of shade on the lower surface of a thread of the screw.

In a similar manner j,j' and m,m' could have been found. $hj—h'j'$, being in front of the meridian plane SI, is visible in vertical projection. $lm—l'm'$ is wholly invisible.

Discussion.

FIRST. The meridian plane, RAK, which is tangent to the helicoid at A,D′, makes an angle of 90° with the horizontal plane. At a point, analogous to any point as e, but on the helix which is at an infinite distance from the axis, the tangent eL would be infinite, hence HL would then be parallel to eL; that is, AH would be both the line of declivity and the element containing the point of contact of the tangent plane. Hence when the *element* of contact of a plane, tangent to a helicoid, is also the line of declivity of that plane, the *point* of contact is on a helix at an infinite distance from the axis.

SECOND. *To find the helix*, H, *the tangent planes for all points of which make a given angle*, β, *with the horizontal plane; β being greater than* α. Construct a cone, having the axis A—A′A″ for its axis, and its vertex at the intersection of an element, E, of the helicoid, with the axis, and let all its elements make an angle equal to β with the horizontal plane of projection. Then construct a plane, through the element E, and tangent to this cone. This plane will make an angle β with the horizontal plane, and its point of contact, on E, found as in (56 Const.), will be a point, p, of the required helix. Having the point p, the helix can readily be constructed.

THIRD. Now let β be the angle made by the rays of light with the horizontal plane of projection, and α, as before, the angle made by the elements of the helicoid with the same plane. The angle β may be greater than, equal to, or less than α. In Pl. VII., Fig. 22, β is less than α, and the curves of shade are open curves with infinite branches. If β were equal to α, a tangent plane of rays would be tangent to the helicoid at the inter-

section of the element, AZ″, with the infinitely remote helices, one on each nappe of the helicoid. At the lower of these points, b and q would unite; and at the upper one, l and h would unite. When β is greater than α, find the helix, H, described above, and apply to it the method of (53–2°), but giving to the elements of the auxiliary cone an angle of declivity equal to β. It will then be found, that but one plane *of rays* can be drawn, tangent on the helix H, and that the points q and b will unite on that helix. Also l and h will similarly unite on a similar helix of the upper nappe, and the curves of shade will be closed curves, wholly within the helix H.

FOURTH. If we substitute for the triangular generatrix of the thread of the screw, a square, having one of its sides in an element of the cylindrical core of the screw (52), a square-threaded screw will be formed, the upper and lower surfaces of whose threads—the axis being vertical—will be right helicoids, having the horizontal plane for their plane director. For all practical cases the angle β is not 0°, hence for the case of the curve of shade on the square threaded screw, β is greater than α, and the curve of shade would therefore be closed. Usually, also, β, and the rate of ascension of the threads, have such values that the curve of shade would be wholly within the core; hence, practically, there is no problem of the curve of shade on the helicoidal surfaces of the square threaded screw.

The elements of shade on its cylindrical surfaces are readily found.

58. Either of the *direct general methods* (48, 50) may be applied in the construction of curves of shade on warped surfaces, but owing to the ease with which a plane is drawn tangent to a warped surface of the second order, the following *indirect general method* will be more convenient.

Observing that a hyperbolic paraboloid may be more easily represented in any position than a warped hyperboloid, *make any element of the given warped surface the element of contact of an auxiliary tangent hyperbolic paraboloid. The point of contact of a tangent plane through this element, can easily be found, and it will be the point of contact on both surfaces.* (Des. Geom. 343.)

In illustration of this method the following problem is given.

Problem XXI.

To construct points of the curve of shade of a conoid.

Principles.—Let the curved directrix of the conoid lie in the horizontal plane, let its rectilinear directrix be perpendicular to the vertical plane of projection, and let this plane be taken as the plane director of the conoid, and of the elements of the first generation of the auxiliary paraboloid. Let the directrices of the first generation of the paraboloid be the line of striction of the conoid, and a tangent to its base at the foot of the assumed element of the conoid, through which a plane of rays shall be drawn. The horizontal plane of projection will then be the plane director of the elements of the second generation of the paraboloid.

Construction.—Pl. VIII., Fig. 24. MEK is the curved, and AA″—A′ is the rectilinear directrix of the conoid. Let TE—T′E′ be the element of the conoid, on which it is proposed to find a point of the curve of shade. Through any point, as A″, A′, of this element, draw the ray A″R—A′R′. This ray pierces the horizontal plane of projection at R, and the assumed element pierces it at E, E′, hence ERG is the horizontal trace of the plane of rays through TE—T′E′. G*h*′, its vertical trace, is parallel to E′T′, since that element is parallel to the vertical plane of projection.

Now, making AF tangent to MEK at E, it follows that AA″—A′ and AF are the directrices of the elements of the first generation of the auxiliary paraboloid, which is tangent to the conoid along the element ET—E′T′.

F*h*—FA′*h*′ is another element of the paraboloid, and it is cut by the plane of rays, through ET—E′T′, at *h*′*h*, which is, therefore, a point in an element of the second generation. But the horizontal plane of projection is the plane director of such elements, AF being one of them; hence *h*′T′ is the vertical projection of the element, and T′ the vertical projection of its intersection with ET—E′T′. Projecting T′ at T, we have T, T′ as the intersection of elements, one of each generation, of the paraboloid. These elements, being contained in the same plane, viz. the plane of rays, T, T′ is the point of tangency of this plane of rays with the paraboloid. But, as the paraboloid and conoid are tangent to each other throughout the element ET—E′T′, the point T, T′ is the

point of tangency of the plane of rays with the conoid, also; and hence is a point of the required curve of shade of the conoid.

Remarks.—a. If the direction of the vertical trace, G′h′, of the plane of rays, were unknown, it would be found by drawing, through any point of ET—E′T′, a horizontal line, as A″g—A′g′, parallel to EG, whose intersection, g,g′, with the vertical plane, would be a point of G′h′.

b. The line hT, being a horizontal line of the plane of rays, is parallel to GE, the horizontal trace of that plane. Also, a vertical line at A, being a directrix of the second generation of the paraboloid, hT necessarily meets FE at A. Hence to find T,T′ we have, practically, the following very simple construction. *First*, draw ER. *Second*, through A draw Ah parallel to ER. *Third*, note T, the intersection of Ah and ET, and project it at T′ on E′T′.

Any other points of the curve of shade may be similarly found.

c. It follows, from the constructions now given, that either projection of the curve of shade of a conoid may be found independently of the other.

Discussion.

FIRST.—*Preliminary.* This discussion may be conducted with reference to the right conoid with a circular base, Pl. VIII., Fig. 25, having the vertical plane of projection for its plane director, and the line, AA″—A′, for its line of striction.

Let cR be the ray through the point c,A′ of the element *ec*. Then eR is the trace of the plane of rays through *ec*. By (Rem. *b*) above, note *a*, where the tangent at *e* meets A″A′. Then *at*, parallel to eR, meets *ec* at *t*, the point of shade in the branch A*th*, and determined by the tangent plane of rays eR.

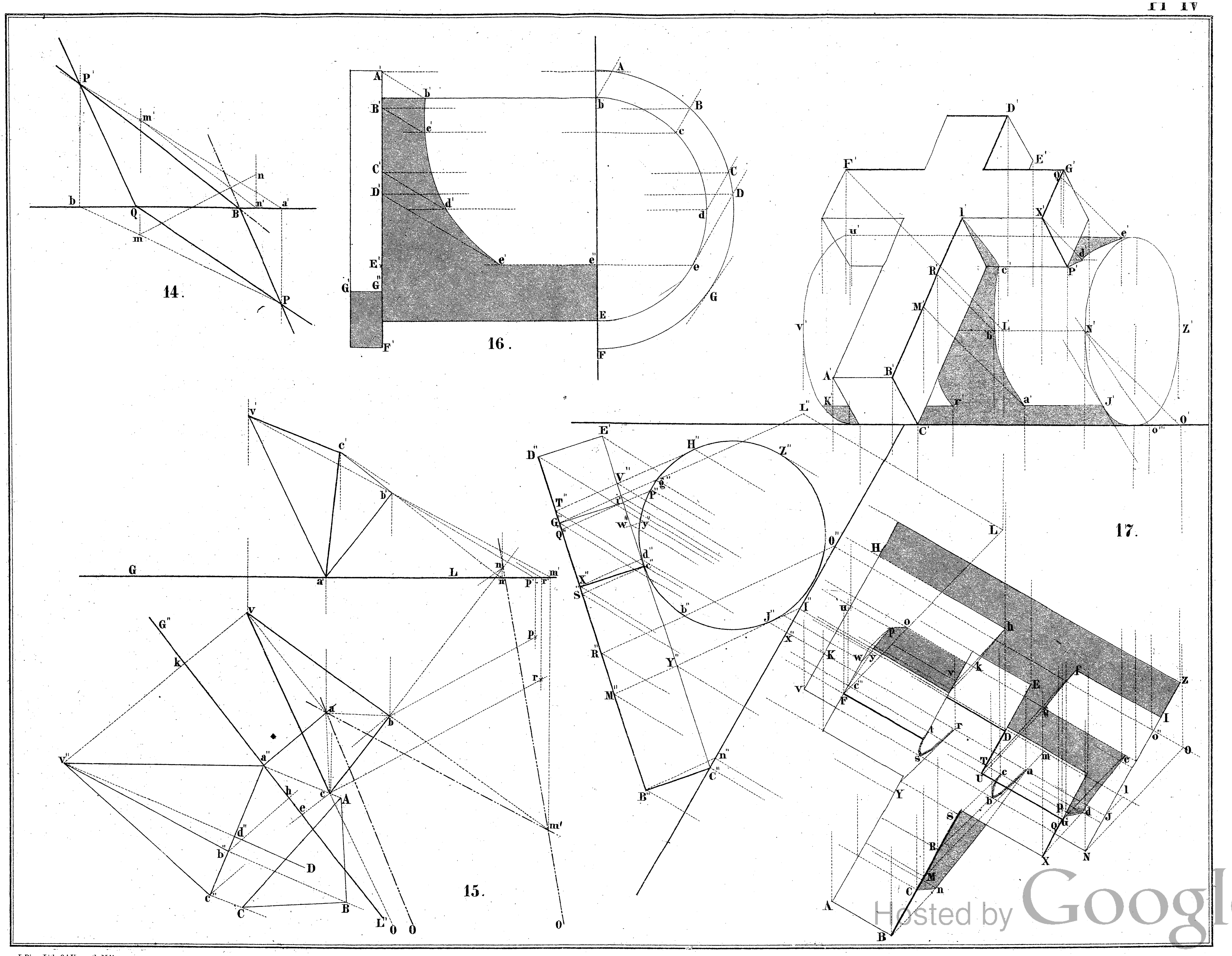
The vertical projection of *t* will be on the vertical projection of *ec*, and in the upper nappe of the conoid. By similar constructions, four branches of the curve of shade may be found, whose horizontal projections are A*th*; AF*m*; A″*k*, and A″*g*.

SECOND. In proceeding with the discussion, the following cases will appear.

First. The horizontal trace of a plane of rays through AA″—A′, may intersect the base of the conoid.

1*st.* In any manner, as at EE″, Pl. VIII., Fig. 25.

2*d.* In the diameter AA″—N′.



J. Bien, Lith. 24 Vesey St. N.Y.

Second. This trace may be tangent to the base of the conoid.

Third. The same trace may be wholly exterior to the same base.

1*st.* At a finite distance.

2*d.* At an infinite distance.

THIRD. For the 1*st* form of the *first case*, it appears: 1°. That the points of shade, consecutive with A,A′ and A″,A′, being points of tangency of parallel rays, on elements consecutive with the vertical elements at A and A″; A,A′ and A″,A′ are cusp points of the first species, as seen in horizontal projection.

2°. That the element which pierces the horizontal plane at K,K′ is, in space, an asymptote to the branches A*h*, on the upper, and A″*g*, on the lower nappe of the conoid. For, the tangent at K, analogous to *ea*, will be parallel to A″A, and hence the points *a*′ and *a*″, analogous to *a*, and determined by this tangent, will be at infinite distances, on A A″ in each direction from J. But lines analogous to *at*, and through such points, *a*′,*a*″, and parallel to the trace of the plane of rays through JK—A′K′, would meet HK—H″K′ only at infinitely distant points of shade, analogous to *t*. Likewise, the element which pierces the horizontal plane at H,H′ is an asymptote to A*m* and A″*k*.

3°. The points A,T,T″ and A″ are all vertically projected at A′, hence the branches A*h* and A″*g*, each of which is partly on both nappes of the conoid, form loops, as seen in vertical projection; to which N′T′ and E′R′ are tangent, at the multiple point T′.

4°. All those parts of these curves of shade are real, or would exist on the exterior of a conoidal opaque solid, like the figure, from whose points rays would pierce the horizontal plane of projection, outside of the base of the conoid.

For the 2*d* form of the first case, AA″ and A′N′ would be the projections of a ray of light; and the loops in the vertical projection would disappear.

FOURTH. In the *second case*, the construction of (Fig. 24—Rem. *b*) will give a curve of one branch, in which T and T″ will unite at J, and loops will reappear in vertical projection, having A′N′ and H″K′ for their tangents at the point J,T′.

In this case, the element JK—A′K′ will also be an element of shade, being an element of tangency of a plane of rays, throughout its whole extent. On this element, therefore, the conoid is

said to be a developable single curved surface. Along the vertical elements at A and A″ planes may also be tangent.

Remark.—FIFTH. In the 1*st* form of the *third case*, the loops of the vertical projection will again disappear, and two branches, as seen in horizontal projection, will proceed from A to the right, and two from A″ to the left.

In the 2*d* form of this case, the ray of light will be horizontal. The vertical projection will be of the same form as in the 1*st* form of this case, but in horizontal projection, the two branches from A, and the two from A″, will coincide.

§ II.—Shadows.

59. The fundamental condition for determining a plane, is, that it shall pass through three points taken at pleasure. Equivalent derived conditions are, that it shall contain a point and a straight line; or shall contain one straight line, and shall be parallel to another. All the elements of a cone have a common point, the vertex, hence, as already seen (41 *b*), it is easy to pass a system of planes; each containing a given line, as a ray, and the vertex; and these will be planes of rays, cutting the cone in rectilinear elements. Likewise, all the elements of a cylinder have a common direction; hence, as before seen (41 *a*), a system of planes of rays can be readily passed through the various elements of a cylinder. But let a plane be passed through a given ray, and a given point of a warped surface; only one element would generally pass through that point, and that, generally, would not happen to be in the supposed plane. Again, let a plane be passed through a given ray and parallel to some given line; only one element would generally be parallel to that line, and that element would not commonly happen to be in the plane employed. Hence we have the *Theorem*, that, *as the elements of a warped surface have neither a common point, nor a common direction, no simply arranged system of planes of rays can be made to cut it, each in a right-line element.*

60. Hence (41) in its application to warped surfaces gives the following *general method.*

Pass any secant plane of rays through any point casting a shadow, and, by constructing its intersections with a number of elements, find its curve of intersection with the warped surface.

Then, the intersection of a ray through the given point, with the curve found, will be a required point of shadow.

61. So much construction for each point of shadow, would be extremely tedious, hence we have the following *inverse general method.* Pass a plane of rays through any assumed element of the warped surface, and note the point, P, cut by it, from the line casting the shadow; then, the ray through this point will pierce the assumed element in a point of shadow.

It is evident, however, that, unless the plane of rays happens to be also a projecting plane, the construction of P will generally be as tedious as that of the curve of intersection, in (60). Hence various special methods, soon to be explained, are mostly employed in finding shadows on warped surfaces.

62. Taking up the topics mentioned in shades on warped surfaces, and in the same order, we have, *first, the shadow of the upper base of a warped hyperboloid upon that surface*, the construction of which will be explained in a subsequent problem of double-curved surfaces.

To find the highest and lowest points. Find the intersection of a ray, drawn through the intersection of the meridian plane of rays with the circumference of the upper base, with the meridian curve contained in that plane.

This construction involves an easy application of the general method of (60) whenever the warped hyperboloid is one of *revolution*, with its axis perpendicular to either plane of projection, for then its meridian curve is readily found.

63. *To find intermediate points.* Employ the *special method* of auxiliary shadows (42). Any horizontal plane, P, will intersect the given surface, its axis being vertical, in an ellipse, or circle, C, if the surface be one of revolution. The shadow of the upper base on the plane P will be a curve equal to that base (33), and the intersections of this shadow with C, will be points of shadow falling on the given surface.

64. In Pl. VI., Fig. 21, the shadow of the curve of shade of the hyperbolic paraboloid upon the horizontal plane, is the parabolic curve Lg, formed by the intersection of the traces of the successive tangent planes 7n, etc. L7, equal and parallel to aA, is the shadow of aA, and 7A is the shadow of A—A″A′.

Problem XXII.

To find the several shadows cast by a triangular-threaded screw upon itself.

65. These shadows are:

First. The shadow of the nut.

1*st.* Of any *unknown point* on an edge of shade of the nut.

2*d.* Of some *assumed* point on such an edge.

Second. The shadow of the curves of shade.

1*st.* On an assumed element.

2*d.* On an assumed helix.

3*d.* On another branch of the curve of shade.

Third. The shadow of the outer helix.

1*st.* On any assumed element.

2*d.* On any helix.

3*d.* On any particular element.

Fourth. There may also be associated with the preceding, the shadow of the entire screw and nut, on the horizontal plane of projection.

1*st.* The determination, by inspection, of all the lines casting this shadow.

2*d.* The construction of the shadow of any assumed point, which will cast a shadow on the horizontal plane.

Each of the above topics, including both its principles and construction, so far as these need be separately mentioned, will now be taken up in the above order.

66. *To find the shadow of the nut upon the screw.*

1*st. The shadow of any unknown point of the nut.*

Principles.—This shadow is found on the principle, that *any* line L in a plane of rays, R, through an edge, E, of the nut, will meet the intersection of R with a given surface, in a definite point of shadow of *some* point in the edge, E; though the line be not, itself, a ray.

This, which is an *inverse method* (61) may be called the *method by assumed lines in planes of rays.* It is usually applicable only when the line E is straight, since a plane of rays can usually be passed through a straight line only (6).

Construction.—Pl. VIII., Fig. 23. The edges of the nut, which cast shadows, are the lower ones, projected at JK and KL, and portions of JI and LN″, all of which are vertically projected in

the line J′L′. Beginning with IJ, this edge, produced, pierces the meridian plane of rays OAB at O,O′. Let OA—O′Q be the direction of the light, then Q, the point where the ray OA—O′Q′ meets the axis A—A′A″, is the intersection of this axis with a plane of rays through IJ. Hence, any meridian plane of the screw will cut a straight element from its surface, and a line from the plane of rays. This line will pass through Q, and will intersect the element in the shadow of some point of IJ. Thus, the meridian plane, A*m*, cuts from the upper thread the line *kj*—*k′j′*, and from the plane of rays, the line *m*A—*m′*Q, which intersects the former line at *l′*,*l*, a point of the required shadow.

In like manner, the edge JK meets the meridian plane of rays, OAB, at *a*,*a′*. Then through *a*,*a′* draw a ray, and Q″ will be determined as the intersection of a plane of rays, through JK, with the axis, A—A′A″. The meridian plane, AB″, now cuts, from the edge JK, the point, B″,B‴, and from the plane of rays the line B″A—B‴Q″. This line meets the element ED—E″D‴, cut from the upper thread, at *n‴*,*n″*, the shadow of some point of JL.

Remarks.—*a.* The points, as *e″*,*e‴*, where the shadow leaves the screw, at its intersection with the helices, are all found by the *indirect special method*, called the *method of intersecting shadows;* and which consists of the following easy operations. *Construct the shadow of the line*, L, *casting the shadow*—in this case the edge of the nut—*upon any plane below. Find also, on the same plane, the shadow of* M, *the line containing the required point of shadow*—in this case the outer helix. Then the intersection of the shadow of L with the shadow of M will be a point, through which a ray will meet M in a point, *m*, and L, in a point, *l*; and *m* will be the shadow of *l*. The point *t*,*t′* is thus found.

b. This construction is not fully shown at R,A″, and *e″e‴*. By drawing the rays through these points, we find the points Y,Y′ and *n*,*n′* of which they are the shadows. Similarly, the points which cast *any* ascertained point of shadow, may be found.

c. From Y,Y′ the shadow of the nut falls on the next thread below, beginning at *q*,*q′*, the intersection of the ray through Y,Y′ with the shadow of the outer helix.

d. The point *e*,*e′* is the shadow of *a*,*a′*, and is the intersection of the ray *ae*—*a′e′* with the element *bc*—*b″c″*. This one point is found by (41) as a *meridian* plane of rays cuts a straight element from a helicoid.

2*d*. *To find the shadow of an assumed point of the nut.*

Let the point be an angle of the nut, as JJ′. Constructions for the shadow of such a point prove, on examination, to be quite numerous. The following are some of them.

First. By the most simple *direct method.* Pass a vertical plane of rays through J,J′. Its intersection with the surface of the thread, will be a curve, easily formed by joining the points in which the plane intersects several assumed elements of the screw, taken, for convenience, in the supposed vicinity of the desired point of shadow. A ray through J,J′ will meet this curve (not shown) in the required point of shadow W,W′.

Second. By an obvious *indirect method.* Find the shadows of points, which may be known to be on IJ and KJ produced. The indefinite shadows of these edges, found as in (66—1*st*), will intersect at a point, W,W′, which will be the shadow of J,J′.

Third. By the following *indirect method*, interesting in theory, but tedious in application. Make J,J′ the vertex of a cone, generated by the revolution of the ray through JJ′, around a vertical axis at JJ′. The intersection of this cone with the screw, is easily found by joining the points of intersection of several neighboring elements of the screw with the surface of the cone. The intersection of the ray element with this curve, is its intersection with the screw, and therefore is the shadow of J,J′.

Fourth. Let a plane of rays be passed through J,J′, so as to cut a rectilinear element from the screw. That is, as such a plane contains a ray through J,J′, it is really required to pass a plane through a given line, and so as to cut a rectilinear element from the screw. The given ray pierces the horizontal plane, Z‴*w*′, at V. Any line through V may be considered as the horizontal trace—on Z‴*w*′—of a plane of rays through J,J′. The upper surface of the upper thread, produced, intersects the plane, Z‴*w*′, in the spiral *wxyz*, and a line from A to any point of this spiral is the horizontal projection of an element of this thread.

Next, make J,J′ the vertex of a cone, generated by the revolution, about a vertical line at J,J′, of a line, J*i*, which makes an angle with the plane Z‴*w*′, equal to the angle made by the elements of the screw with that plane. The circle, with radius J*i*, and centre J, is the base of this cone, in the plane Z‴*w*′. If now a trace, VS, can be drawn, such that SJ and 5A shall be parallel, then these lines, being the projections of lines of equal

declivity, will be parallel in space. Further, as their traces, S, and 5, are in the trace of the plane of rays, the lines themselves are in that plane, and as 5 is also a point in the trace of the screw, 5A is the element of the screw which is contained in a plane of rays through J,J′.

Now, to find VS, draw a number of trial traces V6, V7, etc., and note the corresponding radius vectors, 6A, 7A, etc. Then draw J8, J4, etc., parallel to 7A, 6A, etc., and note their intersections, 8, 4, etc., with the traces V7, V6, etc. Then the curve, 8, 4, etc., is the locus, or place, of all intersections of radii through J, parallel to radius vectors through A, with the traces through V. Hence S, where this trial curve, or locus, intersects the base, *i*lS, of the cone, is the point from which can be drawn the required trace, SV, of a plane of rays through J,J′, and cutting a rectilinear element, 5A, from the screw. Then W,W′, where the ray, JW—J′W′, meets this element, is the required shadow of J,J′ on the screw.

Remark.—It is now apparent, from the preceding construction, that it reduces itself to this problem of two dimensions. Through a given point (V) in the plane containing a spiral of Archimedes and a circle, it is required to draw a line (VS) intersecting the spiral and the circle at points, from which parallels can be drawn, respectively, to the pole of the spiral, and the centre of the circle.

67. *To find the shadow of the curve of shade on the thread below.*

1*st.* *On any assumed element.*

Principles.—This shadow is found by the *indirect special method* of (66—1*st.* Rem. *a*). Thus, the shadow of the curve of shade upon the horizontal plane, will intersect the shadow, on the same plane, of an assumed element, supposed to contain a point of shadow, in a point, through which, if a ray be drawn, it will intersect both the element and the curve of shade. The former intersection will be the shadow of the latter.

Construction.—Pl. VII., Fig. 22. To find that point of the shadow of the curve of shade, *hj′*—*h′j′*, which falls on the element G″*s*—G′*s′*. Drawing rays, G″X and *s*Y, not shown in the vertical projection, through G″G′, and *s,s′*, the line YX will be found, as the shadow of the element upon the horizontal plane of projection. Similarly, *k*Z is found, for the shadow of *hj′*—*h′j′* on the same plane. These shadows intersect at *t*. Then *tu* is the ray which intersects G″*s*—G′*s′* at *u,u′*, which is the required point of shadow.

Remark.—By producing the ray, that point of $hj—h'j'$, whose shadow is uu', could be found.

2*d*. *On an assumed helix.* This shadow is found in the same way as was the one just explained. Thus: let the outer helix, $soe—s'o'e'$, be the assumed helix. VZ is a portion of its indefinite shadow, found by connecting the shadows of several of the points of the helix, on the horizontal plane. This shadow intersects the shadow, kZ, of the curve of shade, $hj—h'j'$, at Z. Then the ray Z$v—v't'$ meets the helix at v,v' which is the shadow of t' (between h and i, in horizontal projection) on the curve of shade —produced in the present case.

3*d*. *On another branch of the curve of shade.* Find, by (66—1*st*.) a point, y'', of the shadow of the curve of shade $hj—h'j'$ produced upon an indefinite element, as Ae, beyond b on the supposed branch, ab, of the curve of shade. Join this point with the previously found points of shadow, juy'', of $hj—h'j$ upon the screw. The indefinite curve of shadow thus found, will intersect ab in the required point, d,d'.

Remark.—To find how far this curve of shadow of $hj—h'j'$ is real, consider that it begins at j,j', the intersection of $hj—h'j'$ with the inner helix j'G$'a'$, and is limited by the ray through h,h', which meets the shadow, $juy''—j'u'v'$, at $p''p'''$. Beyond $p''p'''$, a real shadow is cast, by the outer helix through hh'.

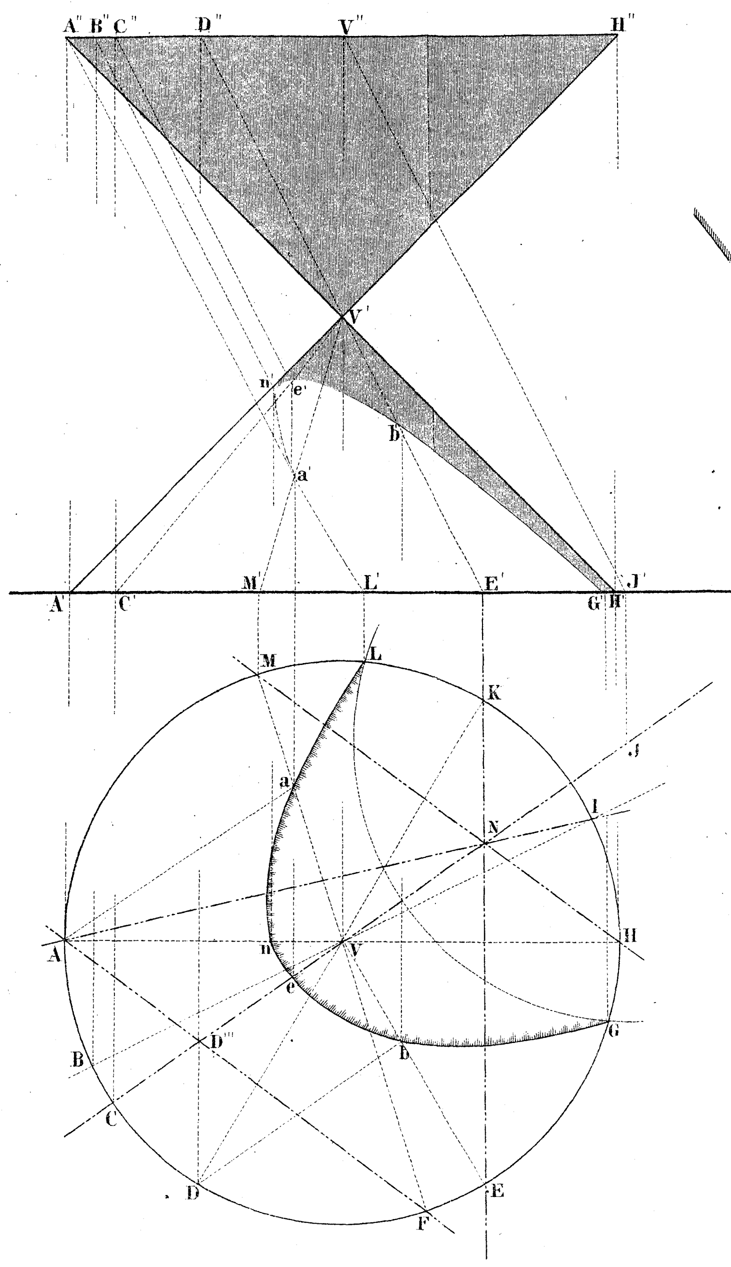
68. *To find the shadow of the outer helix on the upper surface of the thread below.*

1*st*. *On any assumed element. First Method.*—This is the method of (66—1*st*. Rem. *a*) and of (67—1*st*). It is shown, in Pl. VIII., Fig. 23, in the construction of the point f,f'. Here AM—C$''$M$'$ is any assumed element produced, on which a point of the proposed shadow may fall. MP is its shadow on the horizontal plane, and R$''$F is the shadow of a portion of that outer helix, whose vertical projection is $b'''s''$. Then R$''f$, the horizontal projection of a ray through the intersection of these shadows, gives f, which, projected in C$''$M$'$, at f', gives f,f' as the point of shadow required.

Second Method.—This is a *special method* which may be called the *method by helical translation of a plane of rays containing an assumed element.*

Principles.—When a plane is perpendicular to either plane of projection, all lines in it will be projected, on that plane to which the given plane is perpendicular, in the trace of the given plane

18.



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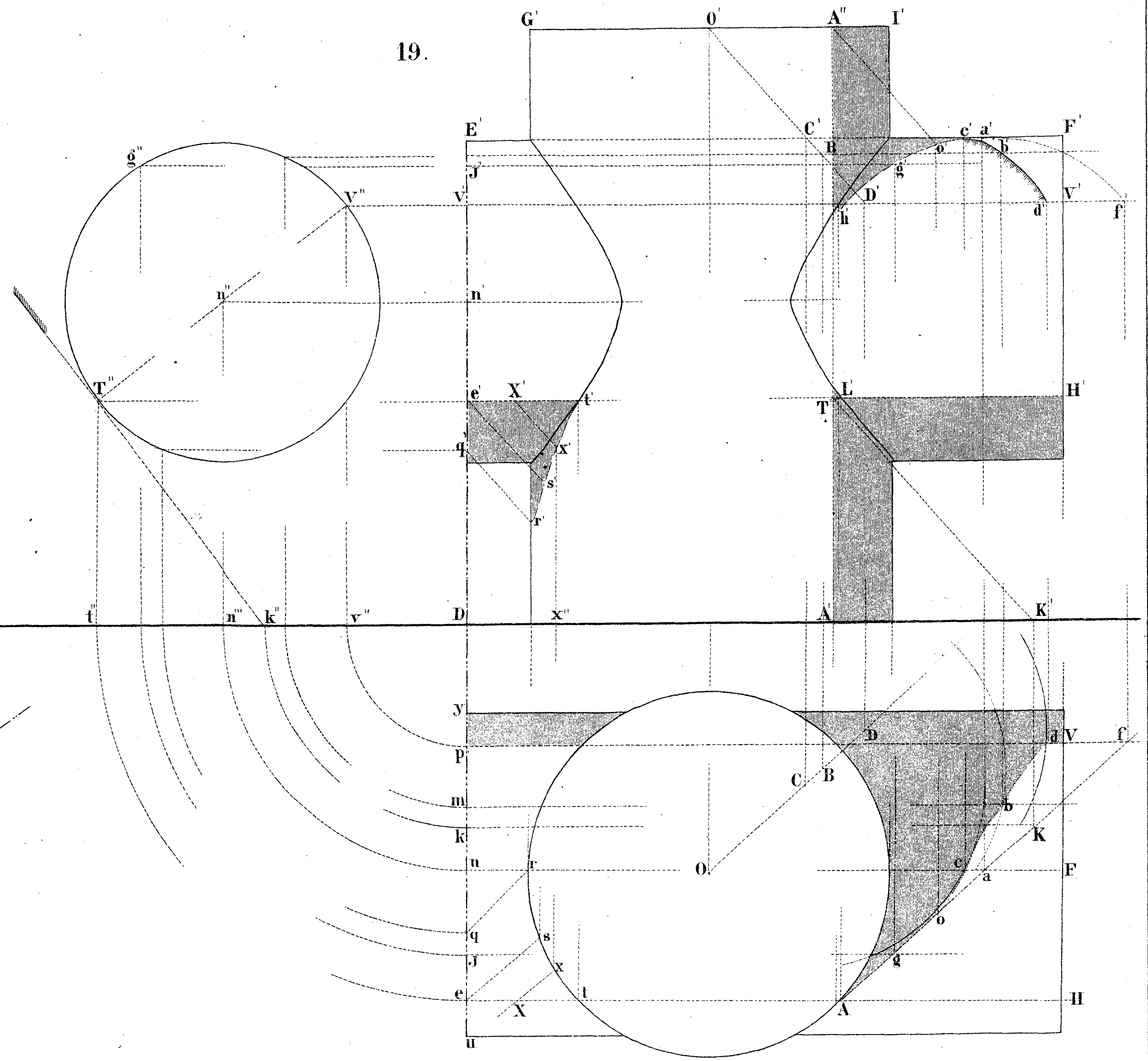

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Also, the intersection of the plane of rays through the element, with the given helix, is the point casting a shadow on that element, at its intersection with a ray through that point.

Construction.—In Pl. VIII., Fig. 23, let AT—I″T′ (53—2°) be any assumed element. It pierces the horizontal plane at N′,N. The ray AB—I″B′, through the upper extremity of the element, pierces the horizontal plane at B′,B. Then NB is the horizontal trace of a plane of rays through the element AN—I″N′. Let this plane now be revolved till it becomes perpendicular to the vertical plane of projection. Its line of declivity, AH, will then be horizontally projected in AH″. Producing AH, the arc CU corresponds to HH″. All points of the plane of rays revolve equally, therefore make T3 = CU, and project 3 at 3′ on the outer helix. All points of the plane also rise equally, and at the rate of the rise of the helix, in order that it may still continue to contain an element of the screw. Therefore make I″I‴ equal to the height of 3′ above T′, and draw I‴3′H‴. As the plane, NB, is now perpendicular to the vertical plane of projection, I‴H‴ is its trace, and it cuts the outer helix at *u*‴*u*″. In the counter revolution of this plane, *u*‴*u*″ traverses a helical arc whose horizontal projection is equal to CU, hence make *u*″*h* = CU, project *h* at *h*′, and then draw the ray *hg*—*h*′*g*′, which intersects the assumed element AT—A″T′ at *g*,*g*′, the required point of shadow.

2*d*. *On any helix.*

This shadow is found by the method of intersecting shadows (67—1*st*). Thus, let the shadow of a spire of the outer helix be found on the spire below, of the same helix. F is the intersection of the shadows, cast on the horizontal plane, by fragments of the outer helixes, near *s*′ and *u*′. Then the ray F*s* gives *s*, whose vertical projection is *s*′. The same ray produced—not shown in horizontal projection—gives *u*′,*u*. Therefore *s*,*s*′ on the outer helix, is the shadow of *u*,*u*′, a point on the spire above. In the same way *d*″,*d*‴ is found.

3*d*. *On any particular element.*

The point *d*,*d*′, in the meridian plane OA, is found by the ordinary *direct method*, as the intersection of a ray, *bd*—*b*‴*d*′, with the element *bc*—*b*′*c*′. 2,2′ may be found either as *f*,*f*′ was, or as *g*,*g*′ was. *sdd*″—*s*′*d*′2′*d*‴ is now the complete shadow of the outer helix, through *b*,*b*‴, on the lower thread. The visible portion of an equal shadow, similarly situated, is shown on the

thread above. On the upper thread, the dotted edge of a similar shadow is shown, as it would appear if the nut were removed.

69. *The shadow of the screw and nut on the horizontal plane.*

1*st. The determination of the lines casting this shadow.*

These lines are, so much of each of the curves of shade as is real, that is, not concealed by a shadow, and on the exterior of the helicoid; so much of each outer helix as divides the illuminated part of the upper surface of a thread from the unilluminated part of the lower surface of the same thread; and, lastly, those edges of shade of the nut, which do not cast shadows on the screw.

2*d. To construct the shadow of any point in any of the above portions of the total line of shade of the screw.*

This construction is made by the simple method of (Prob. III.).

Problem XXIII.

To determine, in general, the shadows of a vertical right conoid.

70. In considering the *shadows of the conoid*, Pl. VIII., Fig. 25, it is to be remembered, that, as all the elements of the conoid intersect the line of striction, AA″—A′, a plane can be passed through this line and any two parallel elements, equidistant from the meridian plane, HK. All such planes will have horizontal traces, parallel to AA″, and intersecting the base of the conoid. A tangent plane along the element JK, is the extreme case of the above group of planes.

From the preceding it follows that any plane of rays, passed through the line of striction, and whose trace, parallel to AA″, is between AA″ and K, as at ER, will intersect the conoid in two rectilinear elements. Of these two elements, the foremost one, as ET, will be the indefinite shadow of the portion, TA, of the line of striction upon the exterior of the conoid. This shadow will be limited by the ray, as A*b*, through A, A′. The opposite parallel element, T″E″, is the imaginary shadow, on the interior of the conoid, of the portion TT″ of the line of striction.

The line A*b* is also the horizontal projection of the imaginary curve of shadow, cast upon the interior of the conoid by the vertical element at A; it being the horizontal projection of the

intersection of a vertical plane of rays, through this element, with the conoid.

When the plane of rays, through the line of striction, does not otherwise intersect the conoid, the shadow of this line will fall wholly on the horizontal plane. In the case just considered, only the portion $T''A''$ casts a shadow on the horizontal plane.

The remaining shadows on the horizontal plane are cast by the curves of shade. They are found by the usual obvious methods.

In this article, only the lower nappe of the conoid has been considered.

DIVISION II.

SHADES AND SHADOWS ON DOUBLE CURVED SURFACES.

§ 1.—Shades.

71. Either a line, or a plane, may be placed tangent to a double curved surface, at a *point* of contact, only. Hence a point in the curve of shade of such a surface, may be conceived of, either as the point of contact of a *tangent ray*, or of a *tangent plane of rays*.

72. A *tangent plane of rays* is, practically, conceived of as containing some given ray, which, with the point of contact, determines that tangent plane. A *tangent ray* is practically conceived of as a tangent to some given, or constructed, section of the given double-curved surface.

73. From the two preceding articles, we have the two following *general methods* for finding points in the curve of shade of any double-curved surface.

First general method.—That of *tangent planes of rays.* By any of the solutions of the problem of orthographic projections, requiring a plane to be drawn through a given line and tangent to a double-curved surface, *construct a tangent plane through any assumed ray. Its point of contact with the given double-curved surface, will be a point in the curve of shade of that surface.*

74. *Second general method.*—That of *tangent rays. Construct any simple section of the given surface, whose plane shall contain a ray of light. Draw a parallel ray, tangent to this section, and its point of contact will be a point of the required curve of shade.*

Remarks.—a. The first of these methods has no formal illustrations among the following problems, numerous special methods being more convenient in the various cases.

b. The second general method is convenient, in finding certain extreme points, as the highest and lowest, on double-curved surfaces *of revolution*. It, therefore, has no separate formal illustrations in the succeeding problems.

c. Each of the following problems is solved by a special method, which is convenient for the purpose, but is not necessarily applicable only to the problem thus solved.

75. *First special method.*—That of *normal planes of shade.* The *essential principle* of this method, which is applicable only to the sphere, is, that the plane of the curve of shade—briefly called the plane of shade—of the sphere, is perpendicular to the direction of the light.

The peculiarity and convenience of this method is, that it requires but one projection of the sphere, as will now be shown.

PROBLEM XXIV.

To find the curve of shade of the sphere, using only one projection of the sphere.

Principles.—Let the vertical projection, only, be given, and let the vertical plane of projection be taken through the centre of the sphere.

Let that plane of rays, which is perpendicular to the vertical plane of projection, be called the perpendicular plane of rays, and let the one, whose angle with the vertical plane equals that made by the light with that plane, be called the oblique plane of rays. Let this oblique plane of rays contain the centre of the sphere.

The oblique plane of rays, and the plane of shade, are perpendicular to each other, and to the perpendicular plane of rays, hence their traces on the latter plane will be perpendicular to each other at the centre of the sphere, and their common vertical trace, T, will be perpendicular to the trace of the perpendicular plane of rays, and will pass through the centre of the sphere.

Now let the perpendicular plane be revolved about its trace, and into the vertical plane of projection. Then consider the great circle, which is the vertical projection of the sphere, as the circle of shade, revolved about its diameter, T, into the vertical plane. Project any of its points into the trace of the perpendicular plane. Revolve it into the trace of the plane of shade on the perpendicular plane. Counter revolve the latter plane to its primitive position, carrying the revolved point. Counter

project the point to the parallel plane from which it originally came.

Construction.—Pl. IX., Fig. 26. LPKR is the vertical projection of the sphere. LK is the vertical projection of a ray, and the vertical trace of the perpendicular plane of rays through the centre of the sphere. O*r*, assumed at pleasure, is the revolved position of a ray, about LK as an axis, and is also the revolved position of the trace of the oblique plane of rays, on the perpendicular plane of rays, LK. OS, perpendicular to *r*O, is the revolved position, about the trace LK as an axis, of the trace of the plane of shade on the plane LK. POR is the common vertical trace of the oblique plane of rays, *r*O, and the plane of shade, OS, on the vertical plane of projection. It is perpendicular to LK, and is the transverse axis of the ellipse into which the circle of shade is projected. LRS is the circle of shade, after revolution about POR, and into the vertical plane of projection. Assume any point *b*, project it, at b', into the trace LK. Revolve it, as at $b'b''$, about O as a centre, into the trace OS of the plane of shade. Counter revolve b'', with the plane LK to b''' in the primitive position of the plane LK, and counter project b''' to B, in the plane *ab* whence it came. Then B will be a point in the required projection of the curve of shade of the given sphere.

All the points of this curve may be found as just described, but it is not necessary to find more than a quadrant, RE*s*, of the curve in this way. For, taking parallel planes, as *ab* and C*h*, equidistant from LK, the point b''' can also be counter projected to C, a point of shade opposite to B from the axis *sk*. Then *n* and B; and *h* and C, etc., will be equidistant from the transverse axis PR.

76. For the practical case in which the projections of a ray make angles of 45° with the ground line, the angle, as *r*OL, made by the ray with the vertical plane of projection, will be 35°–16′ (21). OS, being perpendicular to O*r*, KOS will be an angle of 54°–44′. This angle may easily be constructed, then it will not be necessary to draw O*r*. Thus, take on OK any distance, O*s*, and find the hypothenuse of a right angled triangle, each of whose other sides equals O*s*. On a perpendicular, as *s*S, at *s*, lay off this hypothenuse, then SO, the new hypothenuse thus found, will make the required angle of 54°–44′ with OK, and will therefore be the true position, after revolution of the

plane LK, of the trace of the plane of shade upon the plane LK.

77. *Second special method.* That of the *intersection of the plane of shade with the given double curved surface.* This method can be applied to all the double curved surfaces of the second order (Des. Geom.) for the curve of contact of a cylinder, or cone, of rays, with such a surface, is a plane curve of shade. *This method consists principally in making an auxiliary rectilinear projection of the curve of shade, which may be done by taking an auxiliary vertical plane of projection perpendicular to that plane of shade.*

Problem XXV.

To find the curve of shade on an ellipsoid of revolution.

Principles.—Let the ellipsoid be a prolate spheroid (Des. Geom. 219) and let its principal axis be vertical. Any auxiliary horizontal plane will then cut a horizontal line from the plane of shade and a circle from the ellipsoid.

The two points of intersection of these two lines will be two points in the curve of shade.

Construction.—Pl. IX., Fig. 27. Only half of the ellipsoid is here shown. A—A′D′ is its semi-principal axis. BC—B′C′ is the diameter of its greatest horizontal section. E″F″, parallel to LA, the horizontal projection of a ray of light, is the ground line of a new vertical plane of projection, parallel to the light. The projection of the semi-ellipsoid on this plane is a semi-ellipse equal to B′D′C′. Having noted L and L′, two projections of any point in the ray LA—L′A′, project A at A″, and L at L″, making L″*h*″ equal to L′*h*′. Then L″A″ is the projection of the light on the new plane of projection. Next, draw the tangent ray at T″, and T″A″ will be the auxiliary rectilinear projection of the curve of shade. Now the auxiliary horizontal plane, *m*″*n*″, taken at pleasure, cuts from the plane of shade the line *a*″—*ab*, and from the ellipsoid, the circle *m*″*n*″—*anb*. These intersect at *a*,*a*″ and *b*,*a*″, two points of the curve of shade. On the primitive vertical projection, these points are projected at *a*′ and *b*′, on a line *a*′*b*′, as far from B′C′ as *a*″ is from the ground line E″F.″

Similarly other points are found. A″ is projected at S and V, and then at S′ and V′. T″ is likewise projected at T and T′. Drawing a tangent, parallel to L′A′, we find t',t, on the meridian curve parallel to the vertical plane whose ground line is B′C′.

Only the portion $t'b'$V′C′ of the shade is visible in vertical projection. The boundary of the invisible portion of the shade is dotted.

Remark.—The point T″ is exactly constructed on the principle that the diameter of an ellipse, through the point of tangency, bisects all the chords of the ellipse parallel to the tangent. Hence take any two chords, parallel to L″A″, bisect them, and draw a line through their middle points. Such a line will meet the circumference at T″, the point of tangency of a tangent parallel to L″A″.

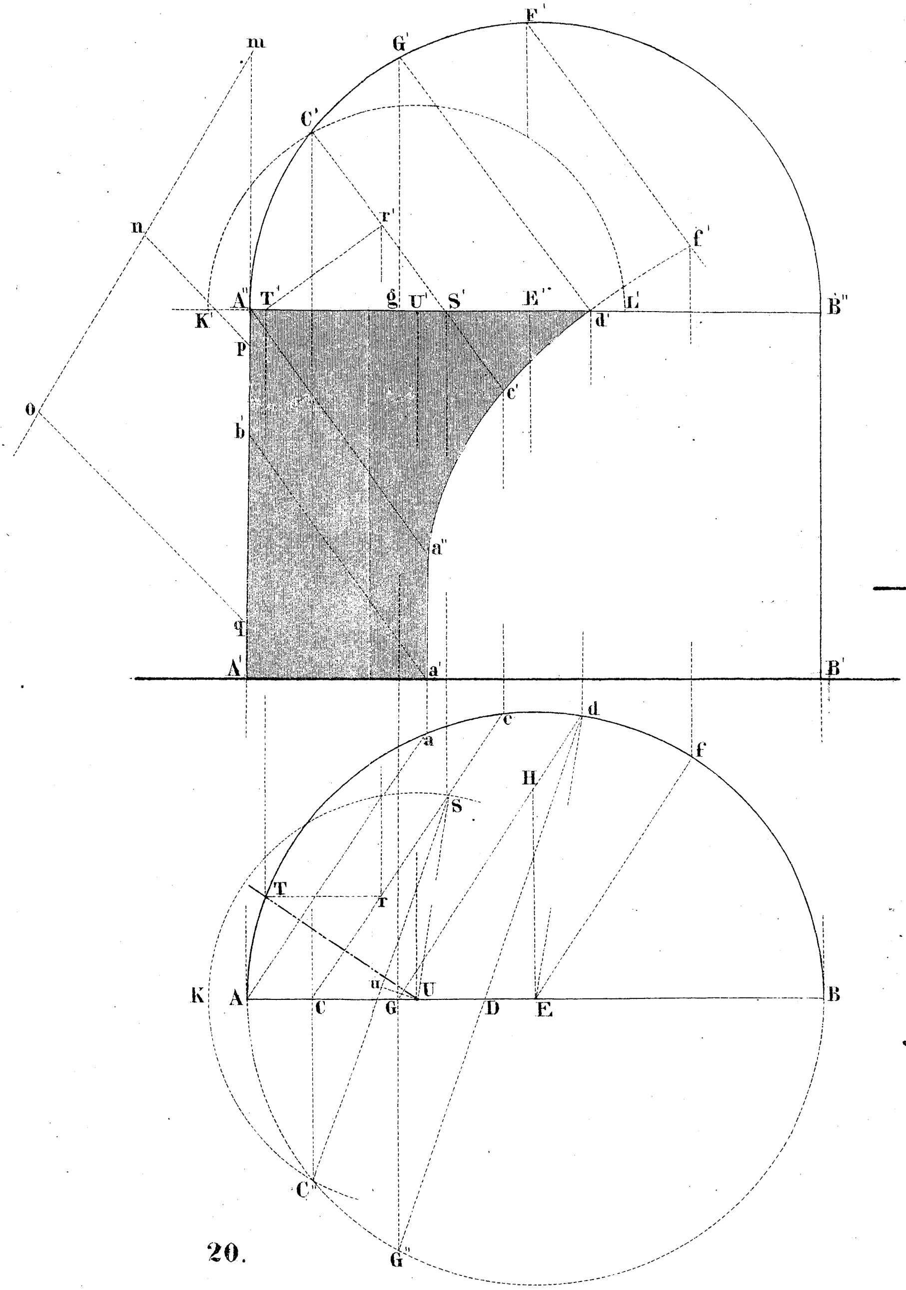
78. *Third special method.* This is essentially the second *general* method (74) but modified so as to be most conveniently applicable to the construction of the curve of shade on an ellipsoid of three unequal axes.

A plane of rays passed through the mean axis, will be perpendicular to the plane of the longest and shortest axes. *Make the ellipse*, E, *cut from the ellipsoid by this plane, the base of a cylinder whose right section shall be a circle. Such a cylinder may be considered as the projecting cylinder of the ellipse upon the plane of right section of the cylinder, taken as a new plane of projection. Other parallel planes of rays will cut ellipses, similar to* E, *from the ellipsoid. Their projections on the new plane of projection will therefore be circles, to which tangent rays may be drawn, which will give projections of points of the curve of shade.*

Problem XXVI.

To find the projections of the curve of shade on an ellipsoid of three unequal axes.

Principles.—Let the longest axis be vertical, and the plane of the longest and shortest axes, parallel to the vertical plane of projection. To locate the projecting cylinder described in (78) make the centre of the ellipsoid the centre of an auxiliary sphere, whose radius shall be equal to the semi-mean axis of the



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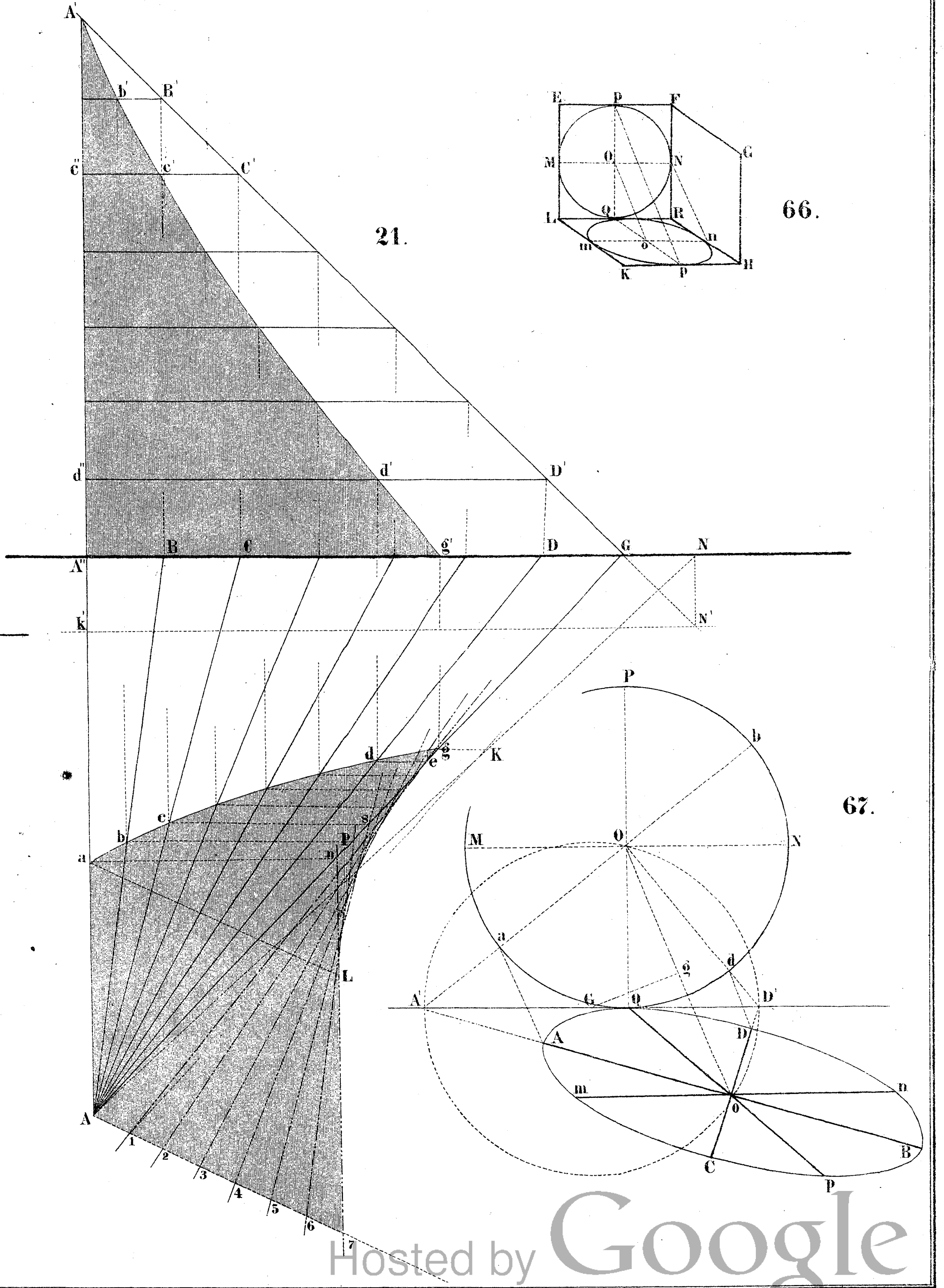

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ellipsoid; then the cylinder will be tangent to this sphere in a circle, which may be considered as a circular projection of the elliptical base of the cylinder.

Any plane, parallel to the circle just described, may be taken as a new horizontal plane of projection, on which a new horizontal projection of a ray of light must be constructed.

Construction.—Pl. IX., Fig. 28. O,O′ is the centre of the ellipsoid. Let O′D′ be the length of the semi-mean axis, whose vertical projection is O′. A circle, with radius O′D′ and centre O′, is the vertical projection, partially shown, of the auxiliary sphere above described.

LO and L′O′ are the projections of a ray of light. Then L′F′ is the vertical trace of a plane of rays, perpendicular, here, to the vertical plane of projection, in being perpendicular to the plane of the longest and shortest axes. J′F′ is the vertical projection of the ellipse contained in this plane of rays, and tangents, from J′ and F′, to the vertical projection of the auxiliary sphere, as at *e*′, are the extreme elements of its projecting cylinder. Take, now, any new ground line, MN, parallel to O′*e*′, and make O′O″ perpendicular to it, and make O″ and O equidistant from their respective ground lines, MN and MP. Likewise make L″ in a perpendicular from L′ to MN, and as far from MN as L is from MP. Then L″O″ is the new horizontal projection of a ray of light. Now the circle with centre O″ and radius O″F (=O′*e*′) is the horizontal projection of the section J′F′, and its diameter *fn*, perpendicular to L″O″, determines precisely the points of contact, *f* and *n*, of rays tangent to the section J′F′—J*n*F*f*. These points being projected at *f*′ and *n*′, give *f*,*f*′ and *n*,*n*′ for two points of the required curve of shade.

By drawing the tangent rays at *t*′ and T′, we find the line *t*′T′, which bisects all the chords, as K′B′, parallel to the tangents at *t*′ and T′. These chords are the vertical projections of ellipses, whose centres, as *m*′, are in the line *t*′T′, and whose horizontal projections are circles having their centres in T*t*, the horizontal projection of *t*′T′.

Thus, *m* is the centre, and *m*K the radius, of the horizontal projection of K′B′, and *kb*, its diameter perpendicular to L″O″, determines the points of shade, *k* and *b*, vertically projected at *k*′ and *b*′. In the same way *g*,*g*′ and *h*,*h*′ are found.

Through the points now found, the projections of the required curve can be sketched; showing that it will be included between

tangents (not shown) as at u and y, tangent also to the vertical projection, $t'f''T'u$, and parallel to $O'O''$.

The horizontal projection of the curve of shade is also tangent, at u and y, to the ellipse, not shown, whose minor axis is Oa—O', and whose longer axis is the distance, on KF, between two lines, perpendicular to it, and tangent to the ellipse $T'C'A'J'$; and the former ellipse is the new horizontal projection of the ellipsoid.

The visible portion of the shade is shaded in vertical projection.

79. *Fourth special method.* This is essentially the special method of (51), but modified so as to be also applicable to the ellipsoid of three unequal axes. It is a property of these ellipsoids—employed in the preceding article—that a set of parallel sections of any one of them, is composed of similar ellipses. Add to this, as just previously shown, that a diameter can obviously be found, in the plane of the longest and shortest axes, which shall be equal to the mean axis. The plane of this diameter and mean axis will evidently intersect the ellipsoid in a circle, and so will all parallel planes. Now by taking a plane of projection—treated most conveniently as a new horizontal plane—and made parallel to these circles, we can readily construct the elements of shade on a series of auxiliary tangent cones having these circles for their bases. The intersection of the elements of shade with the circles will be points in the curve of shade of the ellipsoid.

The construction is left for the student to make.

80. *Fifth special method.* This is the method of *projections of rays on meridian planes.* Its *essential principle* is, that the point of contact of a tangent line to any meridian curve of a surface of revolution, and parallel to the projection of a ray on the plane of that curve, is a point of the curve of shade of that surface. For, a point of shade is the point of contact of a tangent plane of rays (73). But such a plane is perpendicular to the meridian plane through its point of contact. Hence its trace on the meridian plane is the projection of a ray upon that plane, and is also a tangent to the meridian curve, at the point of contact of the tangent plane; that is, at a point of shade; which proves the principle above stated.

The general operations of solution under this method are exemplified as follows:

Problem XXVII.

To construct the curve of shade upon a torus.

Principles.—Conceive of a rectangle, with one of its shorter sides made the diameter of a semi-circle. If the entire plane figure, thus formed, be revolved about the opposite side of the rectangle, the solid generated will be a torus.

If the axis of the torus be supposed to be vertical, the points of shade on its greatest horizontal section will be the points of contact of two parallel and vertical tangent planes of rays. The points of shade on the meridian curve parallel to the vertical plane, will be the points of contact of two planes of rays perpendicular to the vertical plane of projection.

Hence, in finding the four points just described, we make a rudimentary application of the *first general method* (73). Other points of shade are mostly found by the *fifth special method* (80).

Constructions.—1°. *To find the four points on the visible boundaries of the torus.* Pl. IX., Fig. 29. Let ATBt and A′L′B′ be the projections of the torus, and RL—R′L′ a ray of light. Then, by drawing Tt, perpendicular to RL, T and t' will be determined as the horizontal projections of the points of contact of vertical planes of rays, tangent to the greatest circle of the torus at T,T′ and t,t'.

Likewise, at C′ and D′, the centres of the semi-circular ends of the elevation, draw C′n' and D′c', perpendicular to R′L′, and project n' at n, and c' at c, then n,n' and c,c' will be the points of shade on that meridian section of the torus, which is parallel to the vertical plane of projection.

2°. *The highest and lowest points.* These are in the meridian plane of rays KL. Revolve this plane, with the ray RL—R′L′, about a vertical axis at L, till parallel to the vertical plane of projection. Then r''L—R″L′ is the revolved position of the ray RL—R′L′. Through C′ and D′, draw C′u' and D′a' perpendicular to R″L″, and project u' at u and a' at a. Then a,a' and u,u' are the revolved positions of the highest and lowest points. In the counter revolution, a,a' returns to $a''a'''$, and u,u'

to $u''u'''$, which are the highest and lowest points of the required curve of shade. These points are thus found by a simple application of the *second* general method (74).

3°. *To find intermediate points.* Assume any meridian plane, as $d''Le''$, project the ray RL—R′L′ upon it by projecting lines, as Rr—R′r', perpendicular to $d''Le''$. L,L′, being in the assumed plane, is its own projection on that plane, and rL and r'L′ are the projections of the projection of the ray, RL—R′L′, upon the plane $d''Le''$.

If, now, the plane $d''Le''$ be revolved about a vertical axis at L, till it is parallel to the vertical plane of projection, the meridian section contained in it will coincide with the vertical projection of the torus, and the projected ray rL—r'L′ will appear at R′′′′L—R′′′L′. From C′ and D′, draw lines, C′d' and D′e', perpendicular to the revolved ray R′′′′L—R′′′L′, and project d' and e' at d and e. Then, in the counter revolution of the meridian plane, d,d' will return in a horizontal arc to d'',d''', and e,e' to e'',e''', which are points of the required curve of shade.

Other points can be similarly found.

81. The special method of (51) may be applied to any double curved surface of revolution, but for the sake of completeness of illustration, and to show how it would be applied in finding the curve of shade on a warped hyperboloid, it is here applied to a concave double curved surface of revolution, analogous in form to that warped surface. This method will therefore be next described in immediate connexion with the "construction" of the following example:

Problem XXVIII.

To find the curve of shade on a piedouche.

Principles.—A Piedouche, Pl. X., Figs. 30, 31, is a little ornamental pedestal, used for the support of a bust, a statuette, etc.

Conceive of an ellipse, whose axes are in a vertical plane, but oblique to the horizontal plane. Suppose a vertical line, L, in the plane of the ellipse, and so placed as to converge downwards towards its minor axis. Then let that arc of the ellipse which is included between horizontal tangents, and convex towards L, be revolved about L as an axis, and a concave double curved sur-

face of revolution will be formed, which will be the principal surface of the piedouche. The lower base will evidently be larger than the upper, and both may be made the bases of short cylinders having L for their axis. See Pl. X., Fig. 30, which shows how the generating semi-ellipse, AEB, may be formed from a semi-circle, as ACB, where DE, etc. = DC, etc.

The elliptical generatrix may be replaced by a compound curve, composed of circular arcs, tangent to each other, and with different radii.

Remark.—The piedouche formed by the second method will answer every graphical purpose, but is less ornamental, since a circle is a monotonous curve, its points being equidistant from the centre; while the gradually varying distance of the points of an ellipse from its centre gratifies the natural love of variety. While dwelling on a point of taste, it may be added that the generatrix should begin and end on horizontal tangents, because it thus suggests *completeness;* and this, because horizontal boundaries imply *stability;* and this, again, because *gravity*, in reference to whose action the piedouche is to be stable, *acts vertically.*

Constructions.—1°. *Of the piedouche, and of some preliminary points of shade.* Pl. X., Fig. 31. Assume the bases, H′C′ and F′G″, of the double curved portion of the piedouche. At F′, take any vertical height F′E′, greater than half the distance between the assumed bases. Through E′, draw the line E′I′, parallel to the bases, and, from C′, let fall C′D′, perpendicular to E′I′. Then, with E′F′ as a radius, describe the quadrant from F′ to E′I′, and with D′C′ as a radius, describe the quadrant from E′I′ to C′.

The compound curve, thus formed, will be the meridian curve, whose revolution about a vertical axis, A—T′A′, will generate the double curved portion of the piedouche. Its cylindrical bases are, as shown in vertical projection, above and below the double curved portion. These bases are horizontally projected in the concentric circles, A—HYB, and A—GWF.

In applying the method of (51) some of the assumed circles of contact of the auxiliary surface may prove to be beyond the limits of the curve of shade, so that it is best to construct the highest and lowest points of that curve first, as in (Prob. XXVII.).

Let AJ—A′J′ be a projection of a ray of light. Revolve this ray about the axis A—T′A′ to the position AJ″—A′J‴.

Draw I′u' and E′r''' perpendicular to A′J‴, then u' and r''', horizontally projected at u and r'', will be revolved positions of the highest and lowest points. In the counter-revolution, these points return to U,U′ and r,r', which are their real positions.

By drawing I′a' and E′n', perpendicular to A′J′, we find a',a and n',n, the points of tangency of planes of rays perpendicular to the vertical plane of projection. That is, aa' and nn' are the points of shade on the meridian curves forming the vertical projection of the piedouche, and contained in the meridian plane HAF, parallel to the vertical plane of projection. See the *first general method* (73).

2°. *Of intermediate points of shade. First illustration of* (81) *Principles.* Assume any horizontal section of the piedouche—such a section being circular because the axis is vertical—and make it the circle of contact, C, of a tangent cone, whose elements will therefore be tangent to the meridian curves of the piedouche. Drawing planes of rays, tangent to this cone, we find its elements of shade. The two points of intersection of these elements with the circle of contact, C, will be points of shade on the piedouche (51). At the circle of the gorge, the cone will become a tangent cylinder.

Construction.—Assume any horizontal plane, K′L′, Pl. X, Fig. 31, between the highest and lowest points. At either point, as L′, of its intersection with the meridian boundary of the vertical projection of the piedouche, draw the tangent L′S′. This tangent will be an element of the auxiliary tangent cone, whose base is Z′L′.

The circle, with radius Ak, equal to X′L′, is the horizontal projection of the circle of contact, or base, of the cone, and A,S′ is the vertex of this cone. The ray AK—S′K′, through the vertex A,S′, pierces the plane, K′L′, of the base at K′,K. Hence Kk is the horizontal projection of the trace on the plane K′L′, of a plane of rays, tangent to the auxiliary cone. Project k at k'; then, as k,k' is the intersection of the element of shade, Ak—S′k', of the cone, with the circle of contact, Z′L′, it is a point of tangency of the plane of rays, Kk, with the piedouche; that is, k,k' is a point in the curve of shade of the piedouche. But as two planes of rays can be drawn, tangent to the cone, each cone will give two points of shade. Thus, by drawing the chord kt, perpendicular to KA, we find the point of shade, t,t', both of whose projections are invisible.

Likewise, by assuming a circle of contact $b'c'$, whose radius, in horizontal projection, is AQ, equal to half of $b'c'$, and which is the base of a tangent cone whose vertex is W′, we find Q,Q′ and R,R′ for two more points of the curve of shade.

Finally, at the circle of the gorge, $f'm'$, horizontally projected with a radius Ah, equal to half of $f'm'$, the auxiliary cone becomes a cylinder. Hence, by drawing the diameter gh, perpendicular to AK, the direction of the light, we find the two points of shade, g,g' and h,h'.

Remark.—The elements of shade, kA and tA, produced in the direction kA and tA, will meet the base of a cone, tangent to the lower nappe of the piedouche, and having the same inclination that the element L′S′ has, and will therefore give two more points of shade, which can be easily found.

3°. *Second illustration* of the method of (51).

Principles.—Make any circle, between the highest and lowest points of shade, the circle of contact of an auxiliary sphere, tangent to the piedouche internally. The meridian plane, parallel to the vertical plane of projection, and the plane of the assumed circle of contact, which is horizontal, may be taken in this construction as the planes of projection. But the plane of the curve of shade of the sphere is perpendicular to the direction of the light, hence its traces will be perpendicular to the projections of a ray. The vertical trace will pass through the centre of the sphere, and its intersection with that diameter of the circle of contact which lies in the meridian plane just mentioned, is a point of the horizontal trace of the plane of shade of the sphere. The intersections of this horizontal trace with the circumference of the assumed circle of contact, will be two points of the curve of shade of the piedouche, being the points in which the curve of shade of the sphere intersects the circle of contact of the sphere and piedouche (51).

Construction.—Pl. X, Fig. 31. Let M′N′—dNe be the assumed circle of contact. Produce E′N′ to T′ in the axis of the piedouche; then T′N′ is the radius of the auxiliary sphere, whose circle of contact with the piedouche is M′N′. Then T′O′, perpendicular to A′J′, is the vertical trace of the plane of shade of the sphere, and O′,O is a point of its horizontal trace, eOd. The points e,e' and d,d', as explained above, are therefore points of the curve of shade of the piedouche. Through the points now found, the curve can be sketched. It is wholly invisible in horizontal pro-

jection. That part is visible in vertical projection which is in front of the meridian plane GAF.

Remarks.—a. The curve of shade on a sphere may be found as in Prob. XXIV., in Prob. XXV., in Prob. XXVII. Its points may also all be found as are the highest and lowest on the piedouche or torus, since a set of vertical planes of rays would cut the sphere in circles; or they may be found by auxiliary tangent cones, as in (2°.)

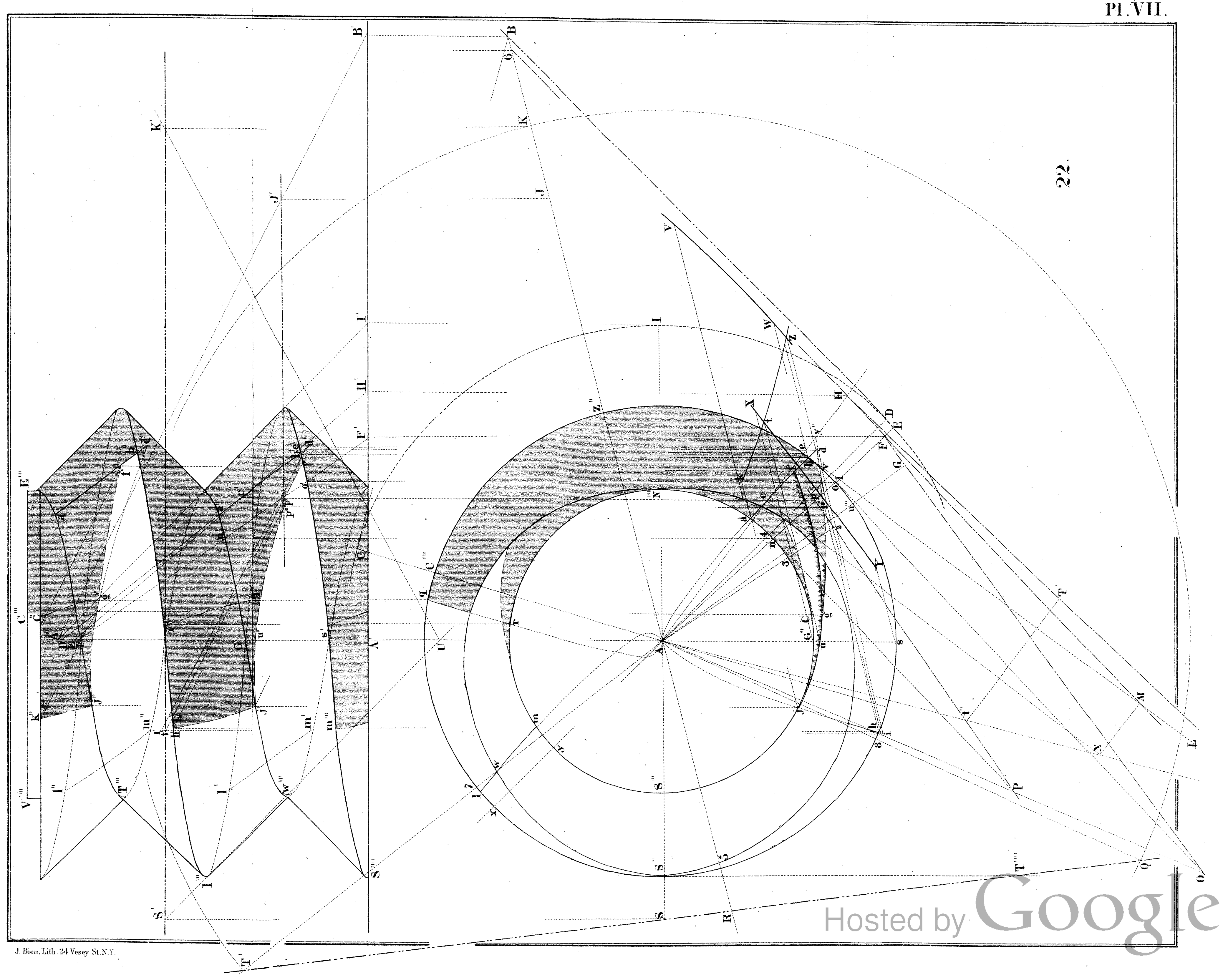
b. Since the piedouche is a surface of revolution, its curve of shade may be found as in the problem of the torus. Likewise the curve of shade of the torus may be found by the methods just employed. In fact, these methods are especially applicable to the interior half of the annular torus, a surface generated by the revolution of a circle about a line exterior to it, but in its plane.

c. By inspection of the figure of the piedouche, the following curious property may be observed. Near the points R,R′, Q,Q′, 1,1′, and 2,2′, tangent rays may be drawn to the projections of the curve of shade at the two projections of the same point. Hence such tangent rays are really tangent to the curve of shade in space. These tangents evidently include the greatest horizontal chords of the curve of shade, one on each nappe of the piedouche.

d. It may be shown in several ways, that there must be such points of maximum width of the curve, at which tangent rays may be drawn to the curve. This may be done, in an elementary manner, as follows. Suppose all points of the curve of shade to be found by the general method of (74) employed in finding the highest and lowest points of shade. A vertical plane of rays near the gorge, will cut a curve similar to A″B″, Pl. X., Fig. 32, to which two tangent rays can be drawn, giving two points of shade, T and T′. Consideration of the form of the piedouche will show, that, as the vertical secant plane recedes from the gorge, the curve, A″B″, will become flatter, the points T and T′ will approach each other, and will finally unite, as at T, on A′B′; T being a point of inflexion, or change of curvature.

If the cutting plane recedes still further from the gorge, the curve will be, as at AB, so flat that no tangent ray can be drawn to it; hence the point T marks the position of the greatest width of the curve.

e. Moreover, the ray is tangent, at T, to the curve of shade itself, for, on that position of A″B″ which is consecutive with



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A′B′, T and T′ will be consecutive points of shade, and their tangents consecutive parallels. On A′B′, therefore, T and T′ unite, and their separate tangents coalesce, forming a tangent to the curve of shade at T.

f. Those of the secant planes which intersect the gorge, cut the piedouche in curves analogous to the meridian curve.

g. Furthermore, the points, as T, Fig. 32, near 1, 2, R, etc., on Pl. X., Fig. 31, are the limits between that part of the curve of shade which really exists on the exterior of a solid opaque piedouche, and on a transparent one, open at the top, and consisting merely of a surface without thickness. All points, as T, *on the upper nappe*, and similar ones at such points at T′, on the curves like A″B″ cut from the *lower nappe*, are imaginary points, on the interior side of the hollow surface. Only such points as T′, on curves shaped like A″B″ and cut from the upper nappe, and such as T, situated on curves cut from the lower nappe, are points in which rays in space are truly tangent to the *external* surface of the piedouche. The construction of a few intersections of the piedouche by vertical planes of rays will make these statements clear.

h. There is a direct construction, given by M. Leroy, of the points, as T, of maximum width of the curve of shade, but it is extremely tedious and comparatively unimportant, being, in the graphical point of view, only approximate. Hence it is not here given. Moreover, it is unimportant, for the further reason that these points are found with sufficient accuracy in sketching the curve by hand through a considerable number of other constructed points.

§ II.—Shadows.

82. The *direct method* of finding shadows on double curved surfaces, is essentially the same as for finding shadows on other surfaces. *Pass a plane of rays through any point*, P, *of the line casting the shadow, and construct the intersection*, I, *of this plane with the given double curved surface. The point, where a ray through* P *meets the line*, I, *is the shadow of* P *on the double curved surface.*

It being desirable, for graphical convenience, that the line I should be a circle, it is evident that the foregoing direct method is always conveniently applicable only to spheres; and to other

surfaces of revolution, only when so situated in reference to the light, that planes of rays may be made to cut them in circles. As these conditions do not generally occur, the shadows on double curved surfaces are mostly found by a variety of indirect and special methods, which will be sufficiently understood from the following problems.

Problem XXIX.

To construct the shadow of the front circle of a niche upon its spherical part.

Principles.—This problem, enunciated more as an abstract one, is this: To construct the shadow of a vertical circle upon the interior of a hemispherical surface of which the given circle is the front edge.

83. If a straight line be moved, so as to remain constantly parallel to its first position, and to rest on the circumferences of two great circles of a sphere, it will generate a cylinder, of which those circles will be equal oblique sections. All the elements of this cylinder will intersect the sphere, except the two which are tangent to it at the extremities of the common diameter of the two great circles. Hence we have the principle, that *a cylinder which intersects a sphere in a great circle, intersects it also in a second great circle.* The right section of this cylinder will be an ellipse, equally inclined to these great circles, and whose transverse axis is their common diameter.

It now follows, that the cylinder of rays passing through the front edge of the quarter sphere, will intersect its interior in an arc of a great circle of that sphere, which arc will be the shadow required.

1°. *The first solution*, based on the foregoing general principles, is made by the *direct general method* (82), and depends immediately on the following—

Principles.—A plane of rays, perpendicular to the vertical plane of projection, will cut a circular arc from the spherical part of the niche, and a ray from the cylinder of rays, whose intersection will be a point of the required shadow.

Construction.—Pl. X., Fig. 33. In this construction, three planes of projection are used.

The plane of the front of the niche is taken as the vertical plane of projection. The plane of the upper base of the niche is taken as the horizontal plane of projection, AC is therefore the ground line for these planes, and the spherical quadrant of the niche is in their second angle. D′H′, parallel to the vertical projection, DO, of a ray of light, is the trace, or ground line, of the auxiliary plane of projection. D′H′ is in the vertical plane, and the auxiliary plane is revolved about it, into the plane of the paper, so that the part in front of the vertical plane falls to the right of D′H′.

DO—D″*p* is a ray of light. When a point is projected on two planes, each perpendicular to a third, its projections on those planes are at equal distances from their respective ground lines, hence, making O′*p*′ equal to O*p*, *p*′ is the auxiliary projection of the point O,*p* of the ray DO—D″*p*. D is projected at D′, therefore D′*p*′ is the projection of the ray, DO—D″*p*, on the auxiliary plane.

The plane of rays, whose vertical trace is DO, cuts from the spherical part of the niche, produced, the semicircle DO*c*—D′T′H′ and from the cylinder of rays (83), the ray, DO—D′*p*′, which pierces the semicircle at *c*′, which, being projected back at *c*, gives *c*,*c*′ as one point of the required shadow—produced.

Likewise, any other parallel plane of rays, as *ak*, cuts from the niche the semicircle *ak*—*a*′*h*′*f*″, and from the cylinder of rays, the ray *ah*—*a*′*h*′, giving *h*,*h*′ for another point of shadow.

As N′T′ is the ray which is the tangent element (83) of the cylinder of rays, T′ is where the shadow begins, and T′*hc* is the quadrant of shadow, on the spherical part of the niche, produced.

2°. *The second Solution* is made by the special method of (42).

Construction.—Pl. X., Fig. 34, CHB—C′A′B′ is the upper base of the semi-cylindrical part of the niche. CAB—C′A″B′ is the front edge of its spherical part.

By the first solution, T,T′, where a plane of rays, N′T′, perpendicular to the vertical plane of projection, is tangent to the spherical part, on its front edge, is the point where the required shadow on the spherical interior begins.

To find another point, take any auxiliary plane, parallel to the front face of the niche. EH may be assumed as the trace of such a plane, on the plane C′A′B′. The ray AE—A′E′, through the centre of the front semicircle, pierces the vertical

plane EH, at E,E′, which is therefore the centre of the shadow of that semicircle on the plane EH. With E′ as a centre, and a radius equal to A′C′ (33), describe the arc which intersects the semicircle H′*h*′, cut from the quarter sphere by the plane EH, at *h*′; whose horizontal projection is *h*, on the trace EH. *h*,*h*′ is therefore a required point of shadow.

The point *y*,*y*′ is similarly found, on a similar vertical secant plane, *a*Y.

3°. *Construction of the point where the shadow leaves the spherical part of the niche.*

This point, *b*,*b*′, may here be found, without reference to the cylindrical part of the niche.

Remembering that the plane of shadow contains a great circle of the spherical part, the diameter A′T′ is its trace on the front face of the niche, and therefore A′A is one point of its trace on the base, CHB, of the spherical part. That trace intersects the semicircle CHB at the point required. Here then is given one trace, A′T′, of a plane, and a point, as *y*,*y*′, in that plane, to find its other trace. Now *ya*—*y*′*a*′ is a line through *y*,*y*′, and parallel to the trace A′T′, and therefore is a line of the plane of shadow. This line pierces the plane C′A′B′ at *a*′,*a*, giving A*ab* for the required trace on C′A′B′, and *b*,*b*′ for the required point at which the shadow leaves the spherical part of the niche; it being the intersection of the plane of shadow with the base line CHB.

4°. *Third Solution.*—This is made by the *special method* of (43) which, in its present application, depends immediately on the following—

Principles.—Knowing, in advance, from (83) that the curve of shadow is a plane curve, we can find its rectilinear projection on a plane of projection, taken perpendicular to the plane of shadow. Then, having this rectilinear projection, both projections of particular points in it can be found as in (43).

Construction.—Pl. X., Fig. 33. D′H′ is the trace, on the plane of the face of the niche, of a plane perpendicular to the plane of shadow. All the planes of projection are, moreover, the same here, as in the first solution.

Having found, as before, *c*′, we have OT′Y for the vertical, and O′*c*′ for the auxiliary, trace of the plane of the shadow, the latter trace being also the projection of the shadow on the auxiliary plane. Any semicircle, as *a*′*h*′*f*″, lying in the spherical

surface, or any ray, as $a'h'$, will cut from the given projection of the shadow a point, as h', whose other projection, h, is found by drawing $h'h$, perpendicular to the ground line D′H′, and noting its intersection, h, with the vertical projection, ak, of the same semicircle, $a'h'f''$, or ray $a'h'$.

Discussion.

First.—If it were not known already that the shadow on the spherical part of the niche is a plane curve, it could be proved by reference to Pl. X., Fig. 33. For, O′D′ and O′c' are equal, hence O′c'D′ is an isosceles triangle. But $a'h'$ being a chord, parallel to D′c', in a semicircle $a'h'f''$, concentric with D′c'H′, it follows that O′$a'h'$ is an isosceles triangle also, and is similar to O′D′c'. Then, as O′a' and O′D′ coincide in direction, O′h' and O′c' likewise coincide, and O′$h'c'$, the projection of the shadow on the auxiliary plane, is a straight line, which shows the shadow itself to be a plane curve.

Second.—Another construction of the point, e, where the shadow leaves the spherical part of the niche, may here be given. C′B″ is the auxiliary projection of a line, perpendicular to the vertical plane of projection at C. Its horizontal projection would be a perpendicular to the ground line, AC, at C; but, to avoid confusion of the figure, a similar line is shown at AB. Producing O′c' to B″, gives B″ as the point in which the plane of shadow intersects C′B″. Now make AB equal to C′B″, and OB will be the horizontal trace of the plane of shadow, after revolving 180° about OU, in the horizontal plane, as an axis. The point m is then the revolved position of the intersection of the horizontal trace of the plane of shadow, with the horizontal semicircle, AUC, of the niche. This intersection is the point sought. Then the horizontal projection, mn, of the vertical arc of counter-revolution, gives n for the primitive horizontal projection of this point, and ne, perpendicular to the ground line, AC, gives e, the vertical projection of the same required point of shadow.

Third.—The point e can again be found, but considered still as the intersection of the horizontal trace of the plane of shadow with the horizontal semicircle, AUC, of the niche. This is done by viewing this trace as the intersection of the plane of shadow with the plane of the horizontal circle AUC. Then, after showing the traces of both of these planes on the planes of projection

whose ground line is D′H′, their line of intersection can be found.

The traces of the plane of shadow are OT′Y, on the vertical plane, and O′B″, on the auxiliary plane D′H′. The traces of the plane of the base of the niche are T′O′Y, on the auxiliary plane, and O′*e*″, on the vertical plane; for, thus considered, D′H′ is supposed to be the trace of the plane DO itself, after being translated, parallel to itself, to the position D′H′, and then revolved, as before described. O′*e*″ is thus parallel to OC, being only a transferred position of OC.

The intersection of the plane of shadow and the plane of the base is now projected in O′*c*′ and O′*e*″. The arc *f*‴*f*″ represents a revolution of the plane of the base, together with the intersection just noted, about O′—T′O′Y as an axis, and into the translated position, D′H′, of the plane of rays DO. The base then appears in D′*c*′H′, and its intersection, O′*e*″—O′*c*′, with the plane of shadow, at O′*f*′. At *e*′,*e*‴ is therefore the revolved position of the point of shadow on the base of the niche. In counter-revolution, *e*′*e*‴ returns to *e*″E, from which is projected, at *e*, the required projection of the required point of shadow. Or, *e*′ may be first counter-projected at *e*⁗, and then counter-revolved, as shown by the arc *e*⁗*e*, which gives, again, the same required point, *e*.

FOURTH.—Considering D′H′, again, as the ground line of the original position of the auxiliary plane of projection, *e* may again be found by the intersection of the projection, O′*c*′, of the shadow on the spherical quadrant, with the auxiliary projection, C′ET′, of the base, AUC, of the niche. C′ET′ is a quarter ellipse on the semi-axes O′T′ and O′C′, C′ being the projection of C.

FIFTH.—A little consideration will show that C′O′ : C′H′ :: E*r* : E*e*′.

Hence O′T′, C′E, and H′*e*′ all meet at I. Hence, from any point, I, on O′O, draw lines to C′ and H′. From the intersection, as *e*′, of IH′ with the semicircle D′*c*′H′, draw a perpendicular to O′O; its intersection, as E, with IC′ will be a point of the ellipse.

Thus, after finding *e*′, as above, we may find E, and hence *e*, by drawing H′*e*′I, and then IC′ which will intersect O′*c*′ at E; which is projected by E*e*, perpendicular to D′H′ at *e*.

Moreover, we have here a construction of the ellipse by homologous secants, analogous to the homologous tangents which may

be drawn at e' and E, and which will meet O′O at a common point.

SIXTH.—Finally, e may be approximately found as the point where the entire quadrant of shadow, *Thc*, on the spherical part produced, crosses AC. (See 42.)

SEVENTH.—*To construct the tangent line at the point*, e.

Principles.—At the point e, the required line is tangent both to the spherical branch, *eh*T′, and to the cylindrical branch of the shadow. For, the tangent, L, to the spherical branch, is the intersection of a tangent plane, P, to the sphere of the niche at e, with the tangent plane, P′, to the cylinder of rays, along the element eE′. The tangent, L′, to the cylindrical branch, is the intersection of the same tangent plane, P′, to the cylinder of rays, along the element eE′, with the tangent plane, P″, along the element, eB″, of the cylinder of the niche. Now, since this cylinder and the sphere are tangent to each other on AUC, the planes, P and P″, coincide, hence the tangent lines, L and L′, coincide also, forming a common tangent to the two curves of shadow which coalesce at e.

Again: The curve *eh*T′ is a plane curve, and the tangent to a plane curve of intersection is the intersection of the plane of that curve, with a tangent plane to the surface on which it lies. Hence, finally, the required tangent line is best found as the intersection of the tangent plane to the cylinder of rays, along the element eE, with the plane of the curve of shadow.

Construction.—Pl. X., Fig. 33. E′Y, tangent to the front of the niche, or base of the cylinder of rays, at E′, is the vertical trace of the tangent plane to the cylinder of rays, along the element eE′. OT′Y is the vertical trace of the plane of shadow. Then Ye, joining the intersection of these traces with the given point e, which is common to both planes, is their intersection, or, the required tangent line.

EIGHTH.—Having now, by previous constructions, the tangents at T′, b', and c, Pl. X., Fig. 34, where c represents the upper end of the vertical shadow of D′C′, it is easy to construct, according to elementary plane geometry, the curved shadow by approximate circular arcs. Thus, from T′ to b' may be made part of an oval of three centres, having A′T′ and A′E′ for its axes; and $b'c$ may be a single arc, tangent to the tangents at b' and c, or, if necessary, it may be an arc of a compound curve having the same tangents, and a horizontal line at c for one axis. When the

projections of the light make an angle of 45° with the ground line, a single arc can be drawn through b' and tangent at T', and another, through b' and tangent at c; which will nearly coincide with the shadow.

PROBLEM XXX.

To find the shadow of the upper circle of a piedouche upon its concave surface.

Principles.—The highest and lowest points of this shadow are found by the *direct general method* (41). Other points are found by the special method of (42).

Construction.—Pl. XI., Fig. 35, shows the surface represented as in Pl. X., Fig. 31. Its projections need not again be described.

1°. *The highest and lowest points of shadow.*

AK is the horizontal trace of a meridian plane of rays. Since the piedouche is a surface of revolution, if we revolve AK about the vertical axis at A, till it becomes parallel to the vertical plane of projection, the meridian curves contained in it will coincide with $E'd'''$ and $B'D'C'$, which are seen in the vertical projection of the piedouche.

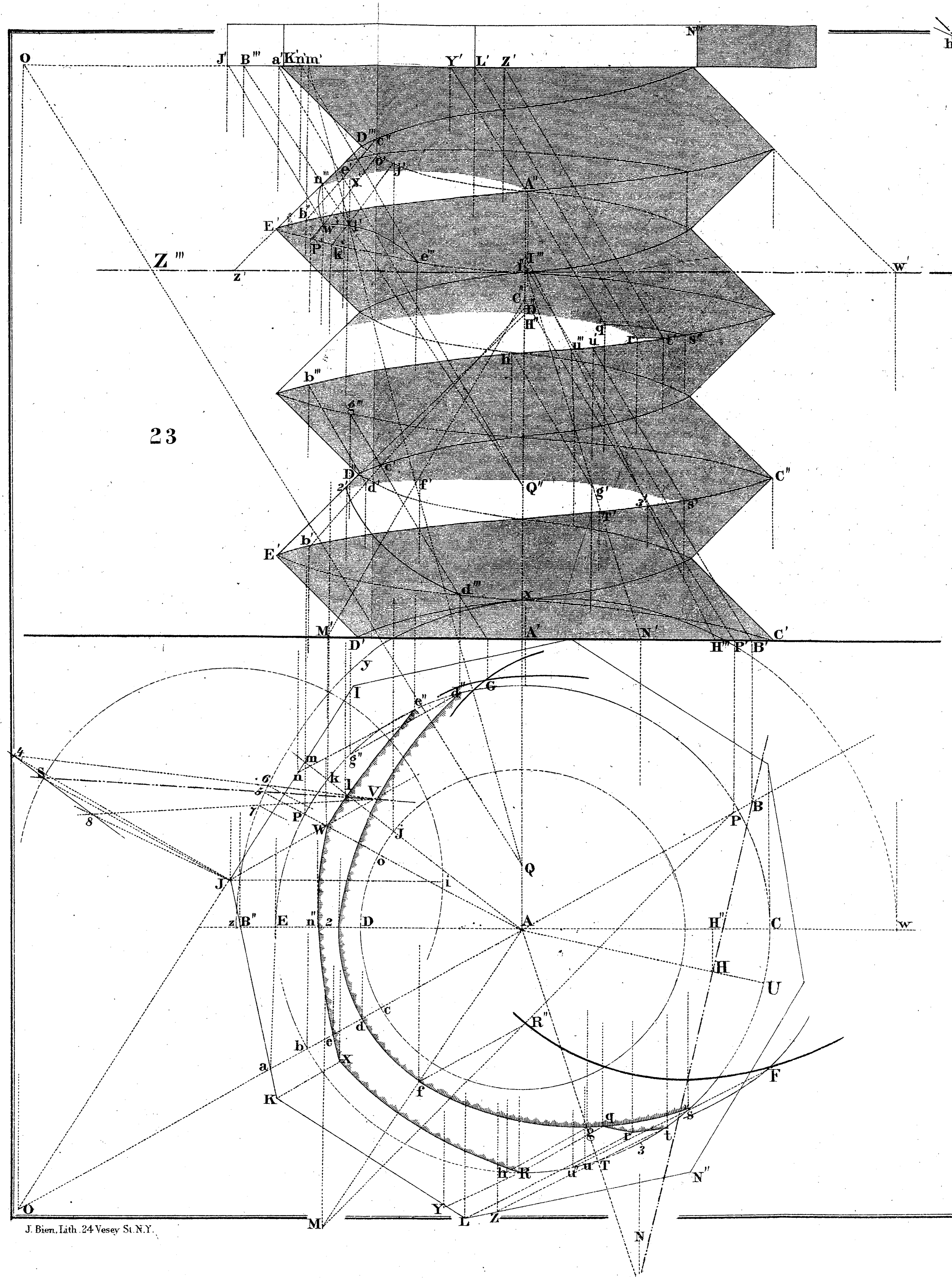
F,F' is the revolved position of the point whose horizontal projection is F'', and which is cut by the plane of rays from the upper circle casting the shadow. Then the revolved position of a ray through F,F', will intersect $F'd'''E'$ and $B'D'C'$, respectively, in the revolved positions of the highest and lowest points of shadow, viz. at d''',d'', and $r'''r''$, in the meridian plane FAC.

The revolved ray, $F'd'''r'''$, is parallel to $A'c'''$, which is found by revolving the ray Ac—$A'c'$ to the position Ac''—$A'c'''$, parallel to the vertical plane of projection.

In the counter-revolution, d'',d''' and r'',r''' return, respectively, in the horizontal arcs, $d''d$—$d'''d'$, and $r''r$—$r'''r'$, to their true positions, at d,d' and $r'''r'$, as the highest and lowest points of the required shadow.

2°. *To find any intermediate points.*

Any horizontal plane, as $q'D'$, cuts from the surface of the piedouche a circle, $q'D'$—qsf; and from the cylinder of rays, whose given base is the circle, BNF—F'A'B', a circle equal to the latter circle, and which is the shadow of BNF—F'A'B' on



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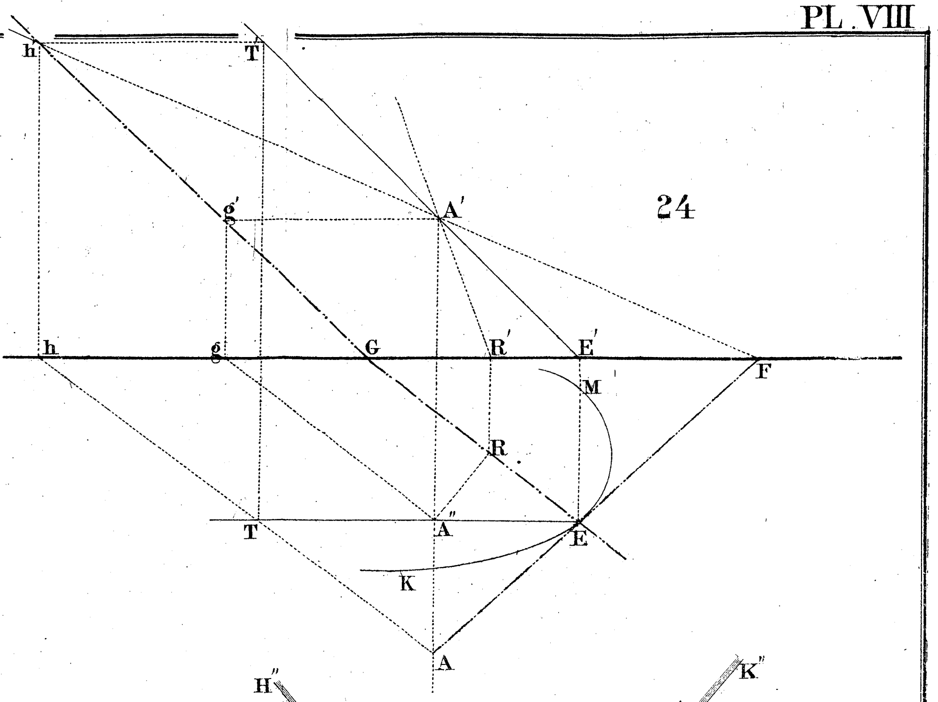

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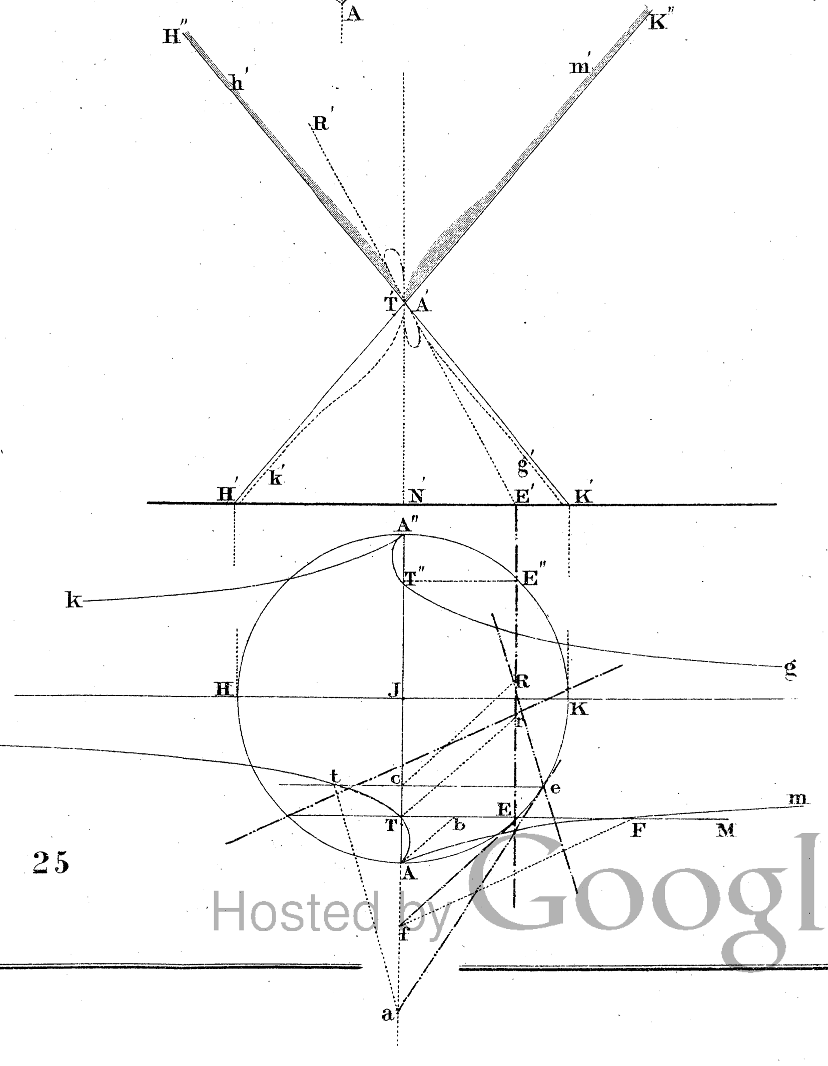

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the plane $q'D'$. The intersection of this auxiliary shadow with the circle qsf—$q'D'$, gives two points of shadow on the given surface (42).

The centre of the auxiliary shadow is b',b, where the ray Ab—$A'b'$, through the centre of the upper circle of the concave surface, pierces the plane $q'D'$. Then, drawing an arc, fq, with a centre, b, and a radius equal to $A'B'$, we find f and q, horizontal projections of two points of the required shadow. Projecting these points on $q'D'$, the vertical trace of the horizontal plane containing them, gives q' and f' for their vertical projections.

Two points can be similarly found, on any other auxiliary horizontal plane between the highest and lowest points, as shown at e,e', on the plane $e'a'$, and at p,p' on the plane $p'c'$.

Discussion.

First.—The whole of the shadow just found, would be real, only in a geometrical sense; or upon a hollow and transparent piedouche. Observing that the highest point of the *curve of shade* would be found by drawing a *tangent* ray, parallel to $A'c'''$, and counter-revolving as before, it is plain that the highest point of *shadow* is *below* the highest point of *shade*. Hence the shadow on an opaque solid piedouche will be real, from its highest point to its intersection with the curve of shade. The real part of the curve of shade was described in Prob. XXVIII. (Rem. *g*). This real part will cast a shadow on the lower part of the piedouche.

Thus the complete line of separation, between the illuminated and the dark portions of the piedouche, consists of the *real* portion of the shadow of the upper circle, the *real part* of the curve of shade, and the shadow of this real shade.

Second.—The latter shadow is thus found. See Pl. XI., Fig. 36, which represents, in horizontal projection, and sufficiently for purposes of explanation, the shadow just mentioned, together with the entire shadow of the piedouche on the horizontal plane.

The shadow of the curve of shade upon the lower part of the piedouche, is found by the special method of two auxiliary intersecting shadows (66–1*st.* Rem. *a*). Thus, *qkt* and *pmu* are shadows of the real parts of the curve of shade, on the horizontal plane. They are found from the projections of those parts,

shown in Pl. X., Fig. 31, and just as any shadow is found on a plane of projection (30).

The circle *sr* is the shadow of an equal circle, UT, near the foot of the piedouche; *r* and *s*, the intersections of these shadows, are the shadows of points of shadow cast by the curve of shade on the circle UT. That is, rays, as *s*U and *r*T, through *s* and *r*, will meet both UT, and the curves of shade. U and T are the shadows, on UT, of the points cut from the curve of shade by these rays.

THIRD.—*pq*, and *ut*, are the shadows of the unreal portions, 1U2, and Q*r*R, Pl. X., Fig. 31, of the curve of shade.

These unreal shadows join the real ones at *p*, *q*, *u*, and *t*, forming cusps of the first order.

This result being apparent, it enables us to learn, as we could not easily do from an inspection of the curve of shade only, in Pl. X., Fig. 31, the real nature of the tangent cylinder of rays, whose curve of contact with the piedouche is the curve of shade.

Considering *putq* as its base, and that its elements are parallel, it is now evident, that above the point 2,2′, for example, its surface curves *upward*, and backward towards the vertical plane; and that below the same point, its surface bends downward, and backward towards the vertical plane; so that it has an edge on the tangent ray at 2,2′. A plane through this ray, or element, and tangent on the same, to both the branches, just described, of the cylinder, is an osculatory plane to the curve of shade at 2,2′; and its horizontal trace is a tangent to the base of the cylinder at *q*, Pl. XI., Fig. 36, between *hq* and *pq*. But the tangent ray at 2,2′, being such a salient edge as just described, a plane section, as the base of the cylinder, will present a cusp as at *q*, where that edge pierces the plane of the section.

Observing, further, that the cylinder of rays, to which the plane of rays is tangent, is, itself, tangent to the piedouche at R, the plane is also tangent to the piedouche at R. Hence its horizontal trace can readily be found.

FOURTH.—The rest of the shadow, Pl. XI., Fig. 36, is thus composed: *ahc* is the shadow of that lower semicircle of the upper base of the piedouche which is towards the light. *deb* is the shadow of the opposite upper semicircle, KSB. *ab* and *cd* are the shadows of the elements of shade of the upper base at K and B. Likewise, *nmy* is the shadow of the upper semicircle,

NHY, of the lower base of the piedouche, and Nn and Yy are the shadows of the elements of shade of the lower base, at N and Y

§ III.—General Problem in Review of Shades and Shadows, determined by Parallel Rays.

84. No better problem can perhaps be found for this review than that of the shades and shadows on the Roman Doric column, embracing, as it does, the shadows both of straight and of curved lines; on planes, and on a variety of single curved, and double curved surfaces.

Description of the Column.—Pl. XI., Fig. 37, represents, without regard to precise architectural proportions, a fragment of the *shaft* and *base* of the column.

Indicating its horizontal circles by their radii, the circle Ca is the plan of the *shaft* a'SK. Next, Cb is the plan of the smallest section of the *scotia* $e'b'c''$. Then Cc is the plan of the *upper* and *middle fillets*, $c'g'$ and $c''h'$. And Cd is the largest section of the *upper torus*, $c''d'h'g'$. Ce—$e'n'$ is the *lower fillet*. CF—$f'n'$ is the *lower torus*. FGH—F'H' is a square member, called the *plinth*. $a'c'g'$ is the *foot of the shaft*, and is a concave double curved surface. The fillets are short vertical cylinders.

Pl. XI., Fig. 38, similarly represents the principal parts of the *capital* and *shaft* of the column. $a''ahw$—$a'b'w'g''h'$ is the *abacus*, whose horizontal sections are squares. The portion, on—$o'n'n''$, of the abacus, is the *cyma reversa*, and is cylindrical. Cg and Cf are the greatest and least circular sections of the half torus $g'g''f'$, called the *echinus*. Cf—$f'f''$ is a *fillet*. Between Cf and Ce is $e'f'f''$, the *cavetto*. Cc is the *neck of the column*, limited by a small torus, not shown, and called the *astragal*, just under which is the *lower fillet* and then a double curved concave surface similar to the foot of the column.

Thus the whole column, between the plinth and the abacus, is a surface of revolution having a vertical axis.

Problem XXXI.

To find the Shades and Shadows of the Shaft and Base of a Roman Doric Column.

Such parts of the solution as present peculiar features, will be figured. Others will be merely referred to previous problems containing similar constructions.

1°. The shaft is strictly a double curved surface, its upper diameter being, in practice, a little less than its lower, and its generatrix slightly curved. The fragments shown in the figures may, however, be treated as vertical cylinders, in finding their elements of shade.

2°. The shadow of the shaft on the foot of the column, is found by the special method of (43) (Prob. XVI., 3°), its horizontal projection being known as a straight line, tangent to the plan of the shaft.

3°. The element of shade, and the upper circle of the fillet, cast shadows on the upper torus. See Pl. XII., Fig. 39.

The *element of shade* is the vertical line at NN′, and its shadow is found as in (2°). Thus, NK is the indefinite horizontal projection of the shadow, NK being the intersection of a vertical tangent plane of rays, at N, with the torus, and therefore parallel to the horizontal projections of rays. *da*—*d′a′* is any assumed horizontal circle of the torus, intersecting the indefinite shadow NK at *a′* whose vertical projection, *a′*, is on *d′a′*, the vertical projection of the assumed circle.

This shadow begins at N′, the foot of the element of shade.

The *upper circle* casts a shadow which is found by the method of (42), see also (Prob. XXX—2°). Thus, the ray, G*h*—G′*h′*, through the centre of the upper circle, pierces an auxiliary horizontal plane, *h′e′*, cutting the torus in a circle, at *h*,*h′*. The circle, *yb*, with centre *h* and radius *hy*, equal to G′*s′*, is the auxiliary shadow of the upper circle, G′*s′*, on the plane *h′e′*. This shadow meets the circle *eb*, cut from the torus by the plane *h′e′*, at two points, one of which is *b*. Then *b* is vertically projected at *b′*, on the vertical projection, *e′h′*, of the circle cut from the torus.

On a complete construction, the latter shadow will intersect the shadow of the element of shade, just before found, and the

curve of shade of the upper torus, in points which the student may construct.

4°. The curve of shade on each torus in found as in Prob. XXVII.

5°. The curve of shade of the upper torus casts a shadow on the middle fillet, which is found as in Prob. IV. But Pl. XII., Fig. 39, serves, however, to show how this shadow can be found, having given but one projection of the curve of shade of the torus. Let $E'o'$ be a fragment of the vertical projection of the curve of shade of the torus. Let o' be a point on this curve, which is supposed to cast a shadow on the fillet. This point is transferred, on a circular section of the torus, to a point whose revolved position is D',D'', found by revolving the semicircular meridian curve, in the plane DGG', about its vertical diameter at S.

In the counter revolution about S, $D'D''$ proceeds to D, in horizontal projection; and in the counter revolution about the vertical axis at G, it returns to o, the desired horizontal projection of o'.

This done, the shadow of o,o' is found by the simple direct method of (40). Thus, op, the horizontal projection of the ray through o,o', pierces the fillet at a point of shadow whose horizontal projection is p. The vertical projection, p', of this point, is at the intersection of the projecting line, pp', with the vertical projection, $o'p'$, of the same ray.

6°. Part of the same curve of shade on the upper torus, casts a shadow on the scotia. Pl. XII., Fig. 40. This curve of shade being of double curvature, its shadow on an auxiliary plane, P as in the special method of (42), will be an irregular curve, a number of whose points must be found, and joined together, to give the auxiliary shadow, whose intersection with the circle cut from the scotia by the plane P, would be a point of the required shadow.

Hence the special method of (42), which, as seen in Prob. XXX., is so convenient when the shadow is cast by a circle, is no more convenient than the direct method (82) when the curve casting the shadow is of double curvature, as in this case.

For this reason the *direct method* (82) as applied in problems like the present, is here used. Let $b'c'$ be a fragment of the vertical projection of the curve of shade of the torus. Let b' be any point on this curve, whose shadow on the scotia is to be

found. It is transferred, on a circular section of the torus, to a', whose horizontal projection is a. In counter revolution about a vertical axis at D; a',a returns to b',b. Now, $b'r'$ is the vertical trace of a plane of rays, taken through b,b', and perpendicular to the vertical plane of projection. It cuts from the scotia the curve efr—$e'f'r'$; which is found, as fully shown, by auxiliary horizontal planes, as $C'f'$. This plane cuts from the scotia the circle $C'f'$—Cf, and from the curve $c'r'$ the point f', whose horizontal projection, f, is on the horizontal projection, Cf, of the circle containing it. Having thus constructed the intersection of the plane of rays with the scotia, by means of a system of auxiliary planes, draw the ray bs—$b's'$, which pierces the curve efr—$e'r'$ at s,s', the required shadow of b,b'. Other points of shadow on the scotia may be similarly found.

The plane of rays may also be taken vertical.

7°. There will be two points of the curve of shade $b'c'$, the rays through which will pass through the points, as e,e', on the lower edge of the fillet. That part of the curve $b'c'$, lying between these points, casts a shadow on the fillet. These points themselves cast shadows on the lower edge of the fillet, and the part of this edge between these points, will cast a shadow on the scotia. The latter shadow will be found as in Prob. XXX.

8°. The curve of shade of the scotia will be found as in Prob. XXVIII.

9°. If, on account of a certain direction of the light, the curve of shade of the upper torus casts a shadow on the lower torus, it will be indicated by the passing of some of the rays, similar to bs, in front of r,r', so as not to cut the corresponding curve similar to efr. This shadow will be found as in (6°).

10°. The shadows of the lower fillet on the lower torus are found as in (3°).

11°. After finding the curve of shade on the lower torus as in Prob. XXVII., its shadow on the top of the plinth will be found, a point at a time, as in Prob. III.

PROBLEM XXXII.

To find the Shades and Shadows of the Capital and Shaft of a Roman Doric Column.

In making the solution of this problem, we have—

1°. The shadow of the upper member of the abacus on the cyma reversa; the element of shade on the latter; its shadow on itself, and on the lower part of the abacus. These shadows form one topic, being all found in the same general way, as seen in Pl. XII., Fig. 41.

The cyma reversa, being bounded by cylindrical surfaces, it is necessary to have an auxiliary elevation of it, as it appears when seen in the direction of the arrow. The point T″ then is the auxiliary vertical projection of the lower front edge, NY—T′A′. Seen in the direction of the arrow, the point, B,B′, of the ray AB—A′B′, appears at a distance, A′*s*, below T″—T′A′, and at a distance, *r*B, to the left of NY—*r*″. Therefore, make T″*r*″ equal to A′*s*′, and *r*″B″ equal to *r*B, then T″B″ will be the auxiliary vertical projection of a ray, or the trace of a plane of rays through NY—T′A′—T″. This plane cuts from the front of the cyma reversa the line *u*″—*uu*′, the upper shadow on the cyma reversa. A parallel, and tangent, plane of rays, L*a*″, gives the element of shade, *n*″—*nn*′, and its shadow, *a*″—*aa*′. Another plane of rays, through the edge below *a*″—*aa*′, would give the shadow on the lower part of the abacus, not shown.

2°. The shadow of the edge *h*′*g*″, Pl. XI., Fig. 38, on all the cylindrical parts below, is found as in Prob. IV.

The shadow of the edge which is perpendicular to the paper at *h*′, on all parts below, is bounded by the intersection of a plane of rays through this edge with those lower surfaces. This intersection will appear in vertical projection as the vertical projection of a ray at *h*′ (43).

3°. The shadow of the lower front edge of the abacus, RT—R′T′, Pl. XII., Fig. 42, on the echinus is found thus: Let N′K′—N*eg* be a circle, cut from the echinus by a horizontal plane, N′K′. A ray, *cd*—*c*′*d*′, through any point, as *c*,*c*′, of the edge casting the shadow, will meet this plane in *d*.*d*′, which determines *dg*, the shadow of RT—R′T′ on the parallel plane, N′K′ (5).

This auxiliary shadow meets the circle, N*eg*—N′K′, at *e*,*e*′ and *g*,*g*′, two points of the required shadow, which is thus found by (42)—see also Prob. XXX.

Observing that *un* is equal to S*t*, the highest point, *f*′, of this shadow, may be thus found. The plane of rays, through RT—R′T′, intersects the meridian plane, D*n*, in a line, *nb*, which is the projection of the light on the plane D*n*. The intersection of this trace *nb* with the echinus is the point *f*′. Revolve *bn* to *ba*″, whose vertical projection is *a*′*b*′. Draw R′L′, parallel to *a*′*b*′, and revolve L to *f*′, and *f*′ is the required highest point of the shadow. By drawing a ray through *f*′, we could find the point whose shadow is *f*′.

The shadow of RT—R′T′ is real, above its intersection with the curve of shade of the echinus, which is found as in Prob. XXVII.

The remaining shadows involve no new operations.

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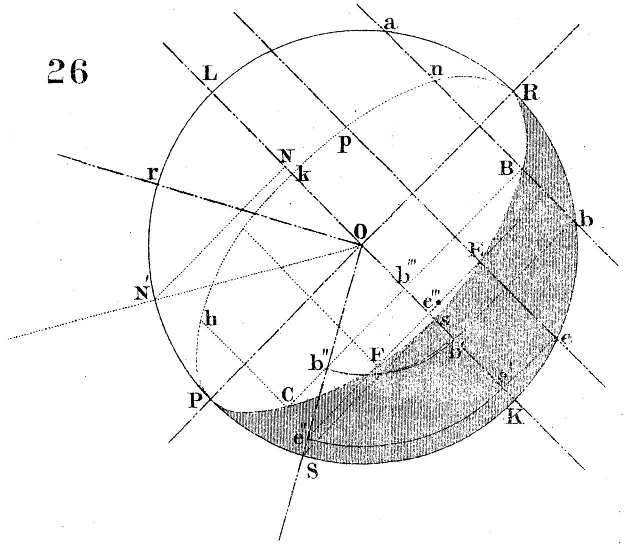

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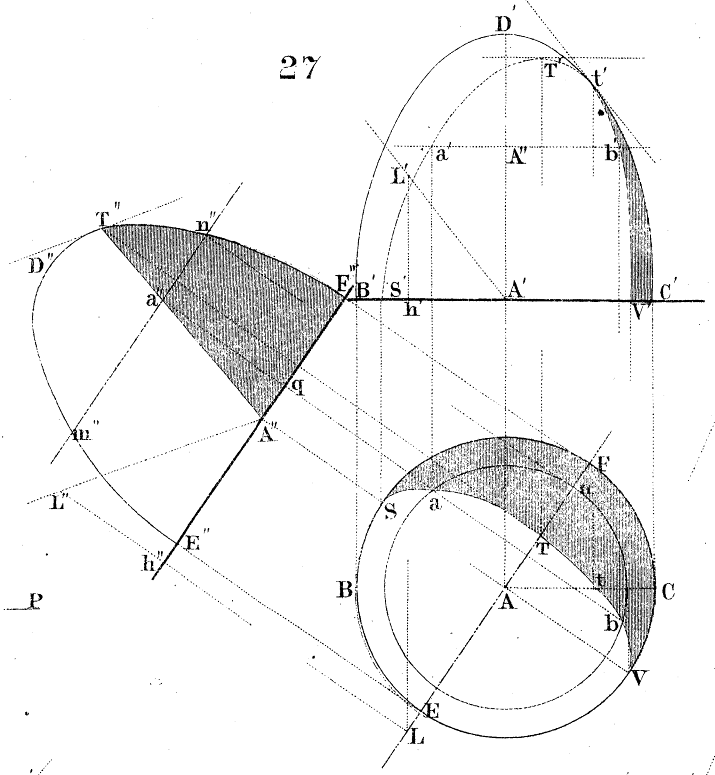

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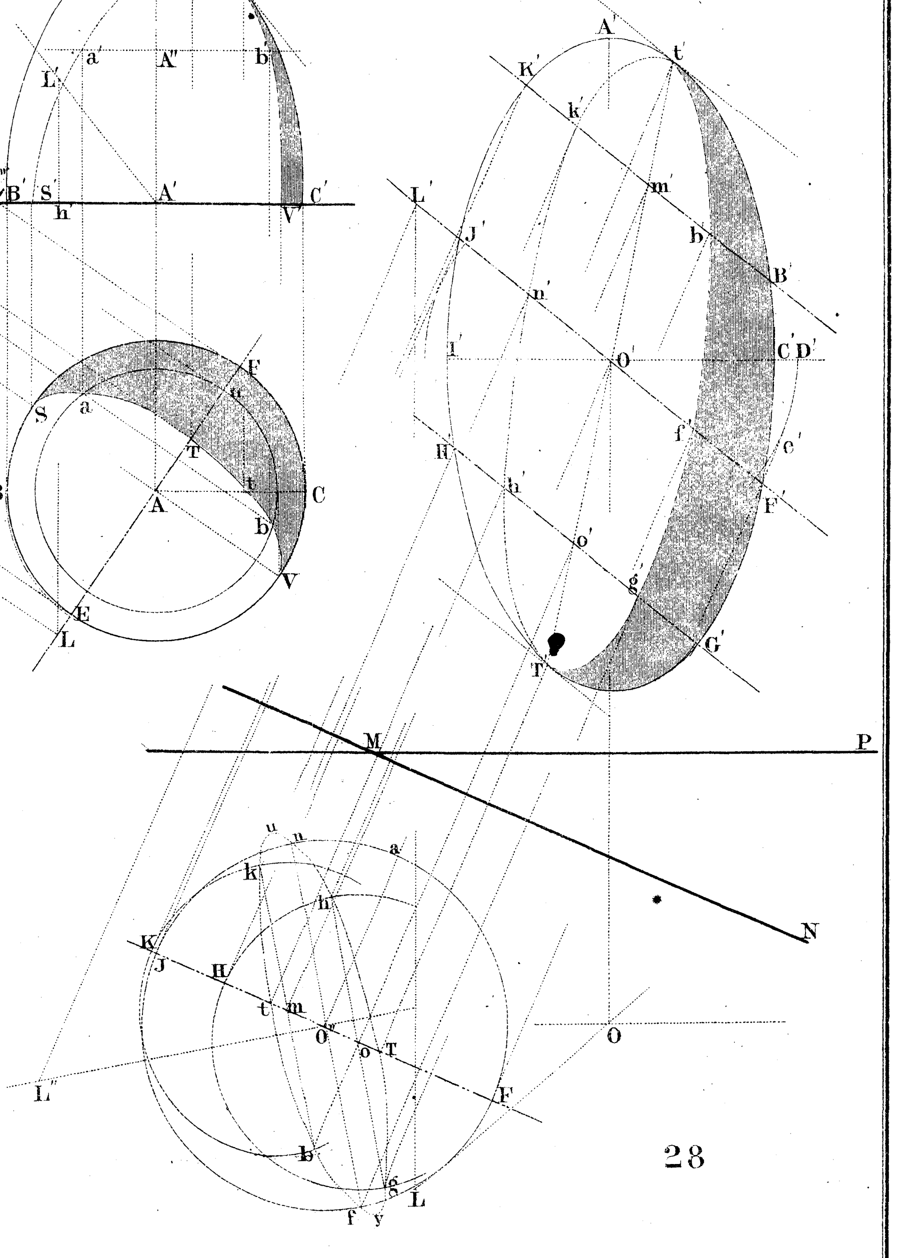

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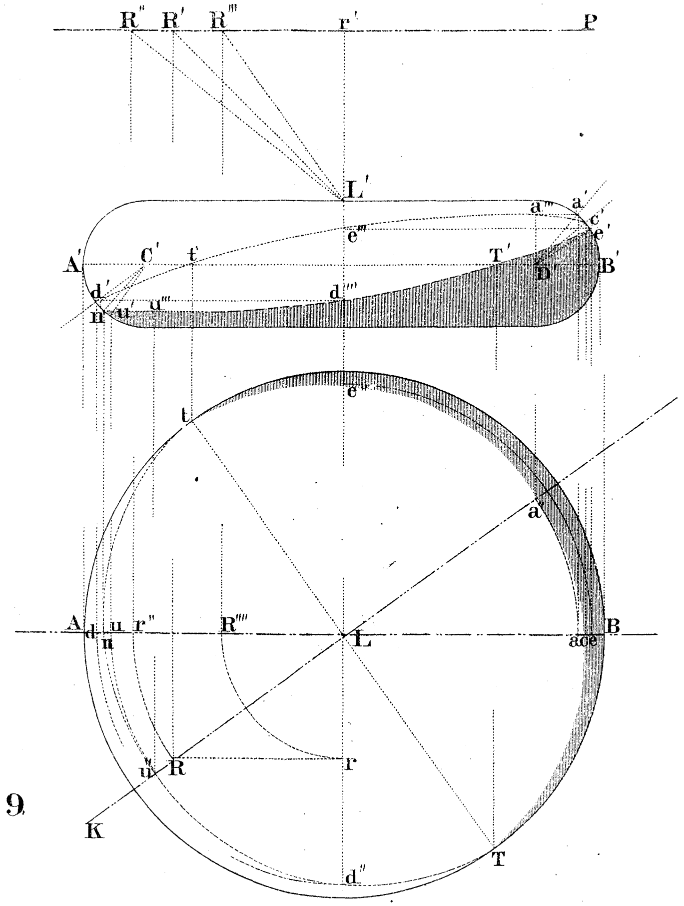

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SERIES II.

SHADOWS DETERMINED BY DIVERGING RAYS.

SECTION I.

General Principles.

85. Having seen, in the preceding general problems, how shades and shadows are found, when the rays of light are parallel, it now remains to examine Arts. (2) and (16) somewhat in detail. When a body, B, is illuminated by a single luminous point, that point may be considered as the vertex of a cone of rays, C, tangent to the body, B. The line of contact of C and B, is the line of shade of B. The interior of the cone, beyond B, is the shadow, in space, of B, and that portion of any secant surface, S, which is within the cone, and beyond B, is the shadow of B on S. If, now, B be illuminated simultaneously by two points, P and P′, a portion, both of B and S, will receive light from neither point. This is the total shade on B, or shadow on S, respectively. Another portion of B and of S will receive light from both points. This will be their completely illumined portion. A third portion of B, and of S, will receive light from one *or* the other of the points, P and P′, but not from both. This is their *penumbra*, or partial shade, or shadow.

86. Let this case be extended to an assemblage of luminous points, forming a luminous body, L. In general, L and B will be of different sizes, hence they may have two common tangent cones of rays inclosing them; one with its vertex beyond both bodies, the other with its vertex between the two bodies. The area, or zone, on B, between the curves of contact of the two cones, will be the partial shade of B. The annular space on S, between the intersections of the two cones with S, will be the penumbra or partial shadow of B on S—the part which is reached by only a part of the rays from L.

87. See, now, Pl. XIII., Fig. 44, where AC represents a luminous body; BD, an opaque body, and PQ an opaque surface which receives the shadow of BD. The dark space, *bd*, is the total shadow, and is bounded by the cone of rays, tangent to both bodies, and having its vertex on the same side of both. The lighter space, *ca*, is the penumbra, and is bounded by the tangent cone whose vertex is L.

88. Regarding L at first as the original source of light, and the vertex of the single cone L*ca*, it may then be regarded as the vertex merely of a complete cone of two nappes, in the outer nappe of which is inscribed the *extended* source of light, AC, around which, and the body BD, may be inscribed the second cone which determines the shadow *bd*.

89. It should be noted, that the volumes of rays will only be cones, when the bodies, AC or BD, are similar, or have similar curves of contact; or when, at least, they can both be inscribed in one given cone, so as to have a continuous curve of contact with that cone.

In other cases, the volumes of rays will be bounded by warped surfaces, but these cases are unimportant, and will not be further considered. In every case whatever, the surface of rays will be a *ruled* surface, since rays of light are straight lines.

90. For the practical case, let L be the sun, and B a body near the earth, S. Here the angular breadth of the penumbra of shade upon a spherical body, B, would be, as found by a simple calculation, only $\frac{1}{107}$ of the radius of B.

See Pl. XIII., Fig. 43, where O is the sun's centre, *a*, the centre of a small spherical body near the earth, and V, and *v*, the vertices of the two tangent cones of rays already described. Then, on account of the relatively great distance of the sun, AB, in the triangle ABC, becomes sensibly equal to his diameter, and *a*C and *ab*, radii of the small body, being perpendicular to BC and AC, respectively, the chord *b*C sensibly reduces to the segment, *b*C, of the side AC, and ABC and *ab*C are similar triangles, and AB : BC :: *b*C : *a*C; or

$$885\,000 : 95\,000\,000 :: bC : aC; \text{ or}$$

$$bC = \frac{aC}{107} \quad \text{very nearly.}$$

This penumbra of shade, whose breadth is *b*C, may therefore be disregarded on terrestrial objects, and the solar rays may, accordingly, be considered parallel.

Remark.—The three following problems, embracing both shades and shadows upon a variety of surfaces, plane, single-curved, and double-curved, are meant to be sufficient to initiate the student in the solution of problems in which the source of light is supposed to be a near luminous point.

After the explanations already made, separate statements of "principles" will not precede the "constructions" of these problems.

SECTION II.

Problems, involving divergent Rays.

PROBLEM XXXIII.

To find the shadow of a semi-cylindrical abacus, upon a vertical plane through its axis.

Pl. XIII., Fig. 45, F*c*E—F′*c*′E′ is the half abacus, with horizontal semi-circular bases, whose diameters are in the vertical plane of projection.

L,L′ is the source of light. A line from this point to any point of an edge, or element, of shade of the abacus, is a ray of light, whose intersection with the vertical plane of projection will be a point of the required shadow.

LD is the horizontal trace of a vertical tangent plane of rays, which determines the element of shade *d*—*d*′*d*″. Then from *d*,*d*″ to E,E′, is the edge of shade of the upper base, and from *d*,*d*′ to F,F′, is the edge of shade of the lower base.

These edges, and the element of shade, cast the line of shadow which is required.

The shadow then begins at F,F′. Any point, as *a*,*a*′, casts a point of shadow A,A′, which is found where the ray L*a*—L′*a*′ pierces the vertical plane of projection.

The element of shade, *d*—*d*′*d*″, casts the shadow D′D″, and the curve D″E′ is the shadow of *d*E—*d*″E′. The shadow of *c*,*c*′, the foremost point of the lower edge of shade, is C′, the lowest point of the shadow; which is now fully determined.

The shadow of a tangent at *c*,*c*′ will be a tangent at C′, parallel to the ground line. Other obvious tangents, useful in sketching the curve, can readily be determined.

Problem XXXIV.

To find the curve of shade on a sphere, the light proceeding from an adjacent point.

Pl. XIII., Fig. 46. Let the centre of the sphere, S$a'dc$, be in the ground line, and let L,L$'$ be the luminous point.

1°. *To find the highest and lowest points of shade.*—LO is the horizontal trace of a vertical plane of rays containing these points. After revolving this plane about the vertical diameter of the sphere, till it coincides with the vertical plane of projection, the luminous point will appear at L$''$,L$'''$, and the great circle cut from the sphere, at S$a'dc$. The revolved rays, L$'''h'''$, L$'''l'''$, tangent to this circle, then determine h''',h'', and l''', the revolved positions of the required points. In the counter revolution, h''',h'' proceeds in the horizontal arc, $h''h$—$h'''h'$ to its true position h,h'. The construction of l,l', the lowest point, will be given presently.

2°. *To find the foremost and hindmost points.*—Go through a series of operations, similar to the foregoing, upon a plane of rays, L$'$O, perpendicular to the vertical plane of projection, beginning by revolving it into the horizontal plane of projection, about the horizontal diameter, whose vertical projection is O. This will give the points f,f', and e,e'. The middle point, n, of $f'e'$, is the vertical projection of the centre of the circle of shade, then l', the vertical projection of the lowest point, is at the intersection of $l'''l'$, the vertical projection of an arc of counter revolution, with the diameter, $h'n'l'$. Then l' is horizontally projected in LO, at l.

3°. *The points on the circles which are in the planes of projection*, are found by drawing tangent planes of rays, perpendicular to the planes of projection. As the centre of the sphere is in the ground line, one circle of the diagram represents both of its projections. Then L$'a'$ and L$'b'$ are the vertical traces of tangent planes of rays perpendicular to the vertical plane. They give a',a and b',b, as the points of shade, on the circle, Sb—S$a'dc$, which is in the vertical plane of projection.

Likewise, Lc and Ld, the horizontal traces of vertical tangent planes of rays, give the points c,c' and d,d' on the circle S$a'dc$—Sb, which is in the horizontal plane of projection.

The construction of other points, as, for instance, those on the great circle, perpendicular to the ground line, is left for the student.

Having now eight points of shade, the curve of shade may be sketched. To avoid confusion, only the vertical projection of the shade is shown, the horizontal projection of the points of the curve of shade being left unconnected. Also, the whole shade on the front hemisphere, is represented as visible, for the sake of clearness, though only a quadrant of the sphere is in the first angle.

Problem XXXV.

Having a niche, whose base is produced, forming a full circle; and a right cone, the centre of whose circular base coincides with the centre of the base of the niche, it is required to find the shades and shadows of this system, when illumined by an adjacent point.

1°. *The shadows on the base and cylindrical part.*

Pl. XIII., Fig. 47. The given magnitudes being familiar ones, their projections, in the position described, may be understood from the figure. L,L′ is the luminous point.

L*a* is the horizontal trace of a vertical plane of rays, through the edge A—A′A″ of the niche, and limits the shadow, A*a*, of the niche upon its base. From *a*,*a*″, the shadow of the same edge is the element, *a*—*a*″*a*′, of the cylindrical part of the niche, limited by the ray L*a*—L′*a*″.

The shadow, *e*,*e*′, of a point, F,F′, of the front circle of the spherical part, upon the cylindrical part, is seen to be found as in Prob. XVIII., the ray being drawn through L,L′.

2°. *The shadow of the spherical part on itself*, is found essentially as in Prob. XXIX., only each point of this shadow requires a separate auxiliary plane, whose vertical trace will pass through L′. Thus, L′N′ is the trace, on the front face of the niche, of an auxiliary plane perpendicular to the vertical plane of projection. The ray, in this plane, meets the semicircle, cut by it from the niche, in a point of shadow. M′*l*″N′, described on M′N′ as a diameter, is the semicircle just named, after revolution about M′N′ into the front face of the niche. The point L,L′ is at a distance, LJ, from the front of the niche, hence, making L′L″ equal to LJ, and perpendicular to L′N′, it results that L″M′*l*″ is the revolved position of a ray, giving *l*″ for the revolved

position of a point of shadow. By counter-revolution about L′N′, the point l'' returns to l', its true position as the shadow of M′.

3°. *The method of* Prob. XXIX. (2*d* solution) *is here shown, also.* Thus, let DE be the horizontal trace of a vertical plane, parallel to the front of the niche. It cuts from the spherical part, the semicircle DE—D′f'E′. The ray, LV—L′O′, through the centre of the front semicircle, AVB, pierces this plane at o,o'. The horizontal line, $o'a'''$, of this plane, limited, at a''', by the ray L′A″a''', is the shadow of the radius O′A″ on the plane DE. Hence $a'''f'$, with centre o', and radius, $o'a'''$, is an arc of the shadow of the front circle, on DE, and f', its intersection with the semicircle D′f'E′, cut from the niche by the plane DE, is a point of shadow on the spherical part.

It will now be easy to find the point n'; the point of which f' is the shadow; and the horizontal projection of the curve T′$l'n'$, remembering that it is a plane curve (83).

4°. *To find the elements of shade of the cone, and its shadows.* Draw the ray LV—LV′ and find U, where it pierces the plane of the base of the niche. Ut and Uq are the horizontal traces of planes of rays, tangent to the cone, and tV—t'V′, and qV—q'V′, are its elements of shade.

The traces Ut and Uq bound the shadow on the base of the niche. At m,m' and p,p' the shadow on the cylindrical part of the niche begins. The ray LV—L′V′ pierces this part at v,v', the shadow of the vertex on the interior of the niche. The shadows of the elements of shade being curves, other points besides m,m' and p,p' must be found. To avoid the acute intersections of projecting lines with q'V′ and t'V′, find, by division, their middle points, giving g,g' and h,h'. Rays, Lg—L′g' and Lh—L′h', through these points, pierce the cylinder of the niche, produced, at s,s' and r,r', respectively. Through these, and the previously found points, the indefinite shadows, $v'm's'$ and $v'p'r'$, of the elements of shade, can be drawn. The points, as u, where the horizontal projections of the rays just drawn meet the horizontal traces of the planes of shade, are the points in which those rays pierce the horizontal plane, and should therefore appear in perpendiculars to the ground line through u', etc., which are their vertical projections.

The niche conceals the horizontal projections of the shades and shadows except the portion of shade VtK on the cone, and tkK, a portion of the shadow of the cone, on the base of the niche.

PART II.

SHADES AND SHADOWS IN ISOMETRICAL PROJECTION.

SECTION I.

General Principles.

91. Isometrical projections of shadows may be found by two quite different methods.

First.—They may be found directly on the isometrical projection of the object which receives them. *Second.*—They may be first constructed in ordinary orthographic projection, and then, from such projections, their isometrical projections may be found. Examples of both of these modes of procedure will be given among the following problems.

92. Since the essential feature of isometrical drawing is, that it shows three dimensions, at right angles to each other, in their real size, it follows, that this kind of drawing is chiefly useful in its applications to rectangular bodies. This fact, also, will be illustrated in the following constructions.

93. In isometrical projection, only one principal plane of projection is used; but, according to first principles, two projections of a line, *i. e.*, its position with respect to at least two planes, are necessary to determine its position in space; hence, in isometrical projection, the rays of light are more conveniently determined by referring them to isometrical surfaces, as those of the body illuminated, than to planes of projection. Hence, again, isometrical drawing is most useful in connection with rectangular objects; since, in representing other objects, auxiliary projections of rays must be employed, as will soon be seen.

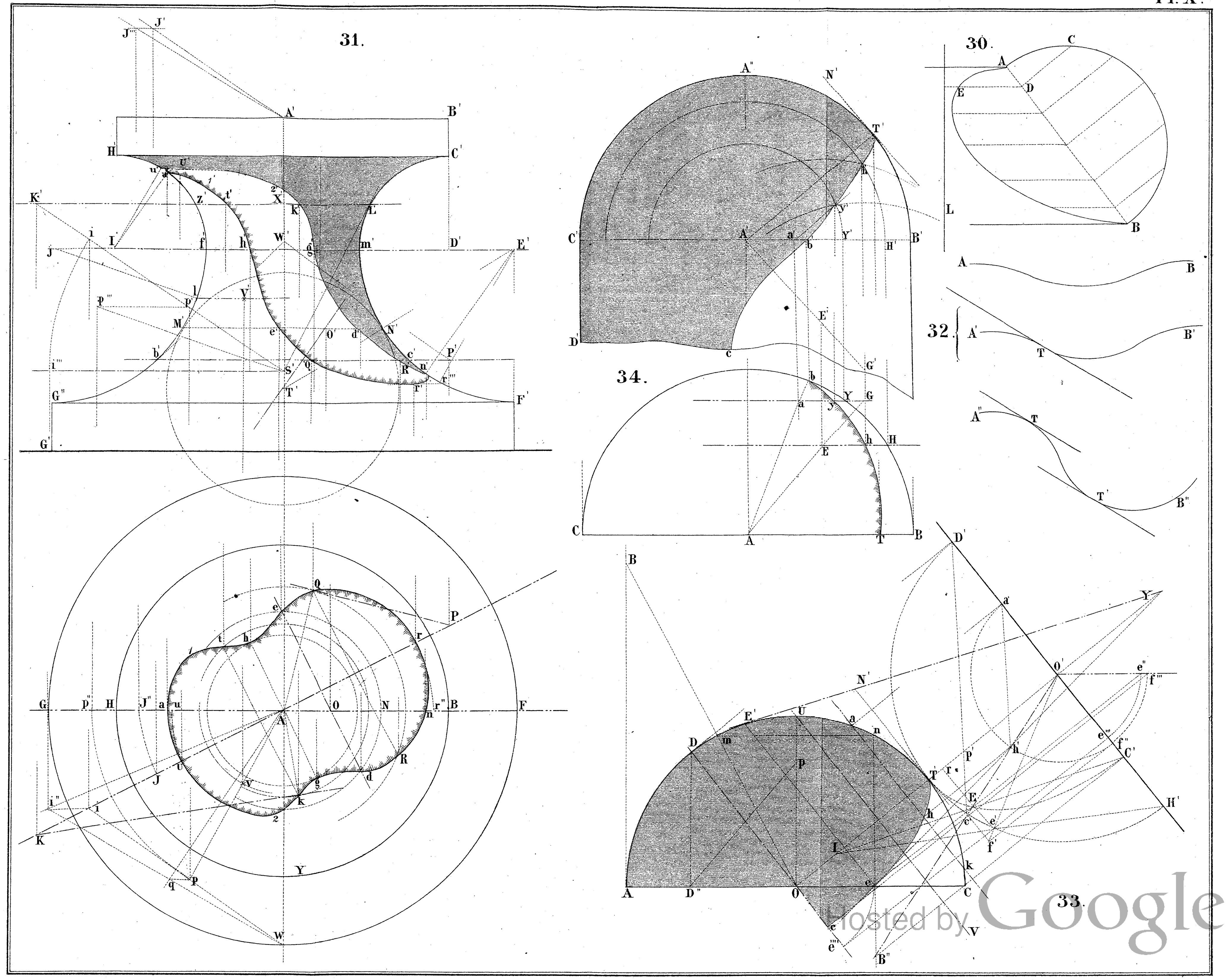
In the following problems only parallel rays are employed. In Pl. XIV., Fig. 48, the light is supposed to enter the cube at the upper left hand corner, E, and to leave it at the lower right hand corner, J, and this is the conventional direction of the light, generally, in isometrical drawing.

The ray, being thus a diagonal of the cube, makes an angle of 35°—16′ with each of its faces. Its projections, as EK and EI, on those faces, are diagonals of the faces, and therefore make angles of 45° with the edges of the cube. In the absence of these planes and edges, as when a curved surface is the subject of a problem, it may be convenient to know the angle made by the ray with the plane of projection. This we next turn to find.

Problem XXXVI.

To find the angle made by the isometrical ray of light with the isometrical plane of projection.

The point, C, Fig. 48, is the projection of the diagonal from the foremost to the hindmost points of the cube, hence the plane of projection is perpendicular to this diagonal. Then, in Fig. 49, let ACBD be the plan of a cube, and D′C′E′G′ its elevation; the cube being in the fourth angle, with its vertical faces making equal angles with the vertical plane of projection, parallel to the paper. Then PQ, perpendicular to the diagonal D′F′, is the trace, on the plane of the paper, of the plane of isometrical projection, and AB and B′E′ are the two projections of the ray seen at EJ, in Fig. 48. Observe that EK, EI, and KI, Fig. 48, are parallel to the isometrical plane of projection. Then B′G′, Fig. 49, is the trace of a plane, parallel to the isometrical plane PQ, and the point A,B′ is its own projection on this plane, and the point B,E′ is projected upon it by the perpendicular E′g, which is seen in its true size. But the diagonals of the cube being equal, D′F′ = the line AB—B′E′, also E′g = $\frac{1}{3}$ of D′F′, and hence of the ray AB—B′E′, and if then the ray, considered as radius, be called 1, E′g = $\frac{1}{3}$ will represent the sine of the angle made by the ray with the parallel plane B′G′ and hence with the isometrical plane PQ. But $\frac{1}{3}$ is the Nat. sine of 19°—28′, very nearly, which is therefore the desired value of the angle made by a ray of light with the plane of isometrical projection.



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SECTION II.

Shades and Shadows on Isometrical Planes.

94. Shadows on isometrical planes will be the intersections of rays, through points casting shadows, with their projections on such planes, or with the traces, on the same planes, of *any* planes of rays containing those points.

PROBLEM XXXVII.

Having given a cube, with thin plates projecting vertically and forward, in the plane of its left-hand back face, to find the shadows of the edges of these plates upon the cube and its base.

Pl. XIV., Fig. 48. *To find the shadow of* AB upon the top of the cube, to which it is parallel.—A*a* and B*b* represent the rays themselves, through A and B; hence, as AE and B*c* are projecting lines, perpendicular to the top of the cube, E*a* and *cb*, on, and parallel to EK, are the projections of these rays, upon the face ECKL. These rays meet their projections at *a* and *b*, the shadows of A and B, hence *ab* is the shadow of AB. E*abc* is now evidently the shadow of the rectangle AB*c*E.

To find the shadow of EDFG.—Here DE, being perpendicular to the face ECIG, its shadow on that face is in the trace, EI, of a plane of rays, through DE, upon that face, and is limited at *d*, by the ray D*d*. From *d*, *dh*, parallel to DF, is the shadow of the portion, DH, of the edge DF. Then *h*F is the shadow of HF on the plane of the lower base of the cube.

We see that Pl. XIV., Fig. 50, shows the case in which the line, AB, casting the shadow, coincides, in projection, with a ray. Either point, as *b*, of the shadow, may be found, either by passing a plane of rays through B, and perpendicular to the top surface FHK, or to the face FKI. The line BD, perpendicular, and the trace D*b* parallel to FI, determine a plane in the former position, and *b*, the intersection of the trace D*b*, and ray B*b*, is the shadow of B. BE, parallel to KH, and the trace E*b* in the direction of EI, Fig. 48, determine a plane in the latter position, above named; and *b* is seen to be, as before, the shadow of B on FKI.

Observe that Eb, and EI, Fig. 48, make angles of 60° with a horizontal line.

SECTION III.

Shades and Shadows on Non-Isometrical Planes.

PROBLEM XXXVIII.

To find the shadow of a hexagonal cupola on a coupled roof, one face of the cupola making equal angles with two adjacent walls of the house.

Pl. XIV., Fig. 51. *To locate the walls and roof.* Let A‴A″Y and YA″n' be portions of the adjacent walls, which are at right angles to each other.

Suppose the inclination of the roof to the horizontal plane, A‴A″n', of the eaves, to be 30°. Find O, the middle point of A‴A″. Then make OA′, horizontal, and equal to OA‴, and make OA′B′=30°, then B′, the intersection of the vertical OB′, with A′B′, is the extremity of the summit of the roof; through it B′A‴, B′A″, and B′M, may be drawn, and through M, the ends of the roof, parallel to A‴B′ and A″B′.

1°. *To locate the cupola.*—Let n be the point at which its axis, nN, pierces the centre line, On, of the plane of the eaves, and let nv be the half width of the square $vgde$, in which the circumscribing circle, $acbe$, of the base of the cupola is inscribed. The semicircle, akb, represents the semicircle, acb, after revolution about the axis ab, till it is parallel to the plane of the paper, *i. e.*, to the isometrical plane of projection. Hence trisect akb, and at k and h draw kc and hd, the projections of the arcs of counter-revolution described by k and h; and ac, cd, and db, will be three sides of the base of the cupola. The remaining sides are parallel to these three. At the corners a, c, etc., of the base, draw the equal vertical edges aC, cI, etc., and join their upper extremities, which will complete the cupola.

2°. *To construct the intersection of the cupola with the roof.*—The plane of the face Icd gives the trace, um, on the plane of the eaves, and the trace uU, on the vertical plane through the ridge B′M. Then Um is its trace on the front roof, and CD, the definite intersection of the face Icd with the front roof. Similar

planes, through Hb, and Ga, give the traces Nn, on the front roof, and NP, on the back roof, and give B and A, as the intersections of these edges with those roofs.

The vertical plane through the ridge, cuts the edges, ac and be, of the base, at s and t. Vertical lines, sS, and tT, from these points, meet the ridge at S and T, where it meets the walls of the cupola. From S and T, draw SA and SC; TB and TE, which, with AF and BD, complete the required intersection.

3°. *To find the shades and shadows.*—The auxiliary planes, just used, being planes of rays, determine the faces whose upper edges are IJ, JH, HK, and KL, as in the shade. Rays, as Ir and Hp, through the extremities of these edges, meet the traces, Cr and Bp, of the planes of rays on the roof, produced, at r, p, etc. Then D$rq'po$QE is the required shadow, of which only the part on the actual roof is real.

The student may reconstruct the figure, with the face ID parallel to the wall YA$''n$.

SECTION IV.

Shades and Shadows on Single Curved Surfaces.

Problem XXXIX.

To find the elements of shade on an inverted hollow right cone, the shadow on its interior, and the shadow of one of its elements of shade on an oblique plane.

1°. *To find the elements of shade.* Pl. XIV., Fig. 52.—Let the isometrical ellipse, SQTN, be the upper base, OV the axis, and the tangents, VQ and VN, the extreme elements of the cone. VOR is a plane of rays, and its trace, OR, on the base, meets the ray VR, parallel to EJ, Fig. 48, at R, the intersection of a ray through the vertex, with the plane of the base. Then RS and RT are the traces, on the plane of the base, of tangent planes of rays, which determine TV and SV as the elements of shade.

2°. *To find the shadow on the interior.*—Rb is the trace of a secant plane of rays, which cuts from the base the point a, casting a shadow, and from the cone, the element, bV, receiving the shadow. Then the ray, aA, determines A, the shadow of a.

In the same way, other points, C, etc., may be found. The shadow ends at S and T. The shadow of *e* is the lowest point of shadow, *e* being the point furthest from the element which receives its shadow. The shadow of *e* is E, on the element *f*V (undistinguishable in the figure from NV). The shadow on the element NV, is found by drawing a trace RN.

3°. *To find the shadow of the front element of shade on an oblique plane*, BLFI.—DKFI is in an assumed horizontal plane through FI; and as its position, relative to the cone, is undetermined by the projection, it is, when produced, assumed as cutting the axis OV at *m*. Then make T*t* equal to O*m*, and *t* is the projection of T on the plane DKFI. Hence *tu* is the trace, on this plane, of a vertical plane of rays through T; *u*U is its trace on the vertical surface BDF, and U*y*, on the oblique plane. The ray, T*y*, therefore gives *y*, as the shadow of T.

Another point of the shadow of TV, is where it pierces the oblique plane. Now *tm*, parallel by construction to TO, is the trace of a meridian plane, OT*tm*, on the horizontal plane DFI. This meridian plane being vertical, it intersects the vertical surfaces, BDF and BDK, in the vertical lines *h*H and *g*G, giving GH for its trace on the oblique plane. But this meridian plane contains, by construction, the element of shade, TV. Hence *r*, where TV meets the trace GH*r*, is the intersection of TV with the oblique plane, produced, and is therefore a point of its shadow on that plane. Hence *ry* is the required shadow of TV.

The shadow of SV may be similarly found.

The distance equal to O*m*, might first be assumed as at *p*P, then P*m*, parallel to *p*O, will locate *m*, and *t* will be the intersection of T*t* with P*t*, parallel to *p*T.

SECTION V.

Shades and Shadows on Double-Curved Surfaces.

PROBLEM XL.

To find the curve of shade on a sphere..

The projection of the sphere on the isometrical plane of projection is the circle CBDA, Pl. XIV., Fig 53. Make the angle KF′E′ equal to PF′C′, Fig. 49, then KF′ will be the intersec-

tion of the plane of the paper, which is the isometrical plane of projection, with an auxiliary plane, taken as a vertical plane of projection, and on which E′F′ is the trace of a plane of rays. Project back O at O′, and make O′E′=O′F′, and each equal to OA. Then E′F′ is the auxiliary projection of that great circle which is contained in a plane of rays. Project E′ and F′ at E and F; then the ellipse on AB and EF as axes, will be the isometrical projection of the great circle just named. Planes of rays, parallel to E′F′, will cut parallel small circles from the sphere, whose diameters will be chords of the horizontal great circle, whose isometrical projection is AHBG.

As these parallel circles will be projected in similar ellipses, assume *gh*, *kn*, etc., as their diameters, and draw *hs* and *nr* parallel to BE, which will give the conjugate axes, *es* and *fr*, of these ellipses. Constructing these ellipses, and drawing rays tangent to them, gives points of shade, *d*, *c*, *b*, etc.

The tangent at *a*, parallel to the rays, is the trace of a plane of rays perpendicular to the plane of isometrical projection, and gives *a* as the point of shade on the great circle, parallel to the paper, which forms the isometrical projection of the sphere.

H and G are points of shade, and a line *a*O*p* is the trace of the plane of shade on a meridian plane parallel to the paper, giving *p* for another point of shade. Other points can be found as first described.

Remark.—For an approximate construction, circles with radii, *eh*, etc., may be employed instead of ellipses, in ordinary constructions.

PROBLEM XLI.

To construct the curve of shade on a torus.

Pl. XIV., Fig. 54. KT is the intersection of the isometrical plane, or plane of the paper, with the auxiliary vertical plane. FL′ is the ground line of the latter plane, on that auxiliary plane which is used as a horizontal plane of projection, and is found by making the angle L′FT=E′F′Q, Fig. 49.

L*da*″ and *a*′*e*′*h*″*u*′ are the auxiliary projections of the torus. The auxiliary projections of the light, in the position which it is desired to have on the isometrical projection, are O″L—R′L′.

We now, according to (91 Second), construct the auxiliary projections of the curve of shade, and then the isometrical projection of the torus and its shade.

1°. *To find four points of the curve of shade.*—From Fig. 48, it appears that the light makes an angle of 35°—16′ with the plane of the base of the cube, *i. e.*, with the horizontal plane of projection. Then, knowing that the horizontal plane (so called in Fig. 54) and seen edgewise before revolution at FL′, makes this same angle with the horizon, it follows that O″L″—R′L‴ is a ray, after being revolved about the (relatively) vertical axis O″—L′O′, till parallel to the auxiliary vertical plane; and R′L‴ is found horizontal. Then draw rays, tangent at *e*′ and *u*′, and parallel at R′L‴, and *e*′ and *u*′ will be the revolved positions of the highest and lowest points of the curve of shade, whose true positions are *h*′ and R′.

The points of shade on the greatest horizontal section of the torus, are *a*″,*a*′ and *b*″,*b*′.

2°. *To find intermediate points.*—Let the points in the meridian plane, *cd*, be found. According to (80) O″*l* and R′*l*′ represent the ray O″L—R′L′ after being projected on this plane, *cd*. Then O″*l*″ and R′*l*‴ are the revolved positions, in the parallel meridian plane, *a*″*b*″, of the projected ray. Next, draw from *r*, the centre of the semicircular part of the meridian curve, *rd*‴, perpendicular to R′*l*‴, then *d*‴,*d*″ is the revolved position, and *d*,*d*′ the primitive position of a point of shade; O″—L′*h*′ being the axis of revolution. For variety of construction, revolve Y*c* to the position Y*c*″, parallel to the vertical plane of projection. This semicircle will then be vertically projected, with the radius *p*′*m*′; and by drawing *p*′*c*‴, perpendicular to R′*l*‴, we find the point of tangency, *c*‴,*c*″, of a revolved ray, parallel to R′*l*‴, and from this, the same point in its true position, *c*,*c*′.

3°. *To construct the isometrical projection of the torus, and of its shade.* Assume *eu* for the trace of the meridian plane, *a*″*b*″, upon the isometrical plane.

The highest points, *e* and *u*, are projected from *e*′ and *u*′ upon this trace. The right and left hand points, E and G, are projected from O′, and EG is the projection of the diameter E″G″. To find intermediate points, as those in the plane *cd*, project the direction of vision, *s*O″—*s*′O′, upon this plane as at SO″—S′O′. Then revolve this projected ray, together with the meridian plane, about a vertical axis at O″ till parallel to the vertical

plane of projection, and then by (80) the points of contact of projecting lines, parallel to $s'''O'$, will be points of apparent contour in the isometrical projection. These points are best found by drawing radii, as rn''', perpendicular to $s'''O'$, which gives $n'''n''$, and after counter-revolution, n,n'. There being evidently four such points, make $bN=ns$, and then making $Oa=Ob$, make aI, aB, and bA, each equal to bN. The oval figure passing through the points now found, will be the isometrical projection of the torus.

For the curve of shade; h' is projected at H, tH being equal to $h'h''$. $Oq=Ot$, and then, $q\text{R}=t\text{H}$. The points c' and d' are then projected at C and D, at distances from eu equal to the distance of d from $a''b''$. Finally, a' and b' are projected in the meridian curve, $a''b''$, whose isometrical projection is eu, at a and b. Through the six points now found, with f and k, the points of tangency of tangent planes of rays perpendicular to the isometrical plane, the curve of shade can be drawn.

H, the highest point, determines the visible portion of the curve of shade to be fHDk.

Remarks.—a. After all the labor of making the foregoing construction, it is now more fully evident, according to (92), that while this, and the three preceding constructions, afford good examples for study, they, and the present problem particularly, show that isometrical projection has little or no advantage in respect to clearness of representation, except as applied to plane-sided bodies having solid right angles, whose sides are equally inclined to the plane of projection.

b. On account of the superior pictorial character of isometrical projections, and still more, of the oblique projections explained in a previous volume (Elementary Projection Drawing) under the name of "Cabinet Projections," it is of less importance to represent shades and shadows upon them. The cabinet projections of shades and shadows can readily be made from their common projections.

Book 2.

THE FINISHED EXECUTION OF SHADES AND SHADOWS.

CHAPTER I.

THEORY AND CONSTRUCTION OF BRILLIANT POINTS; AND OF GRADATIONS OF SHADE.

SECTION I.

Preliminary General Principles.

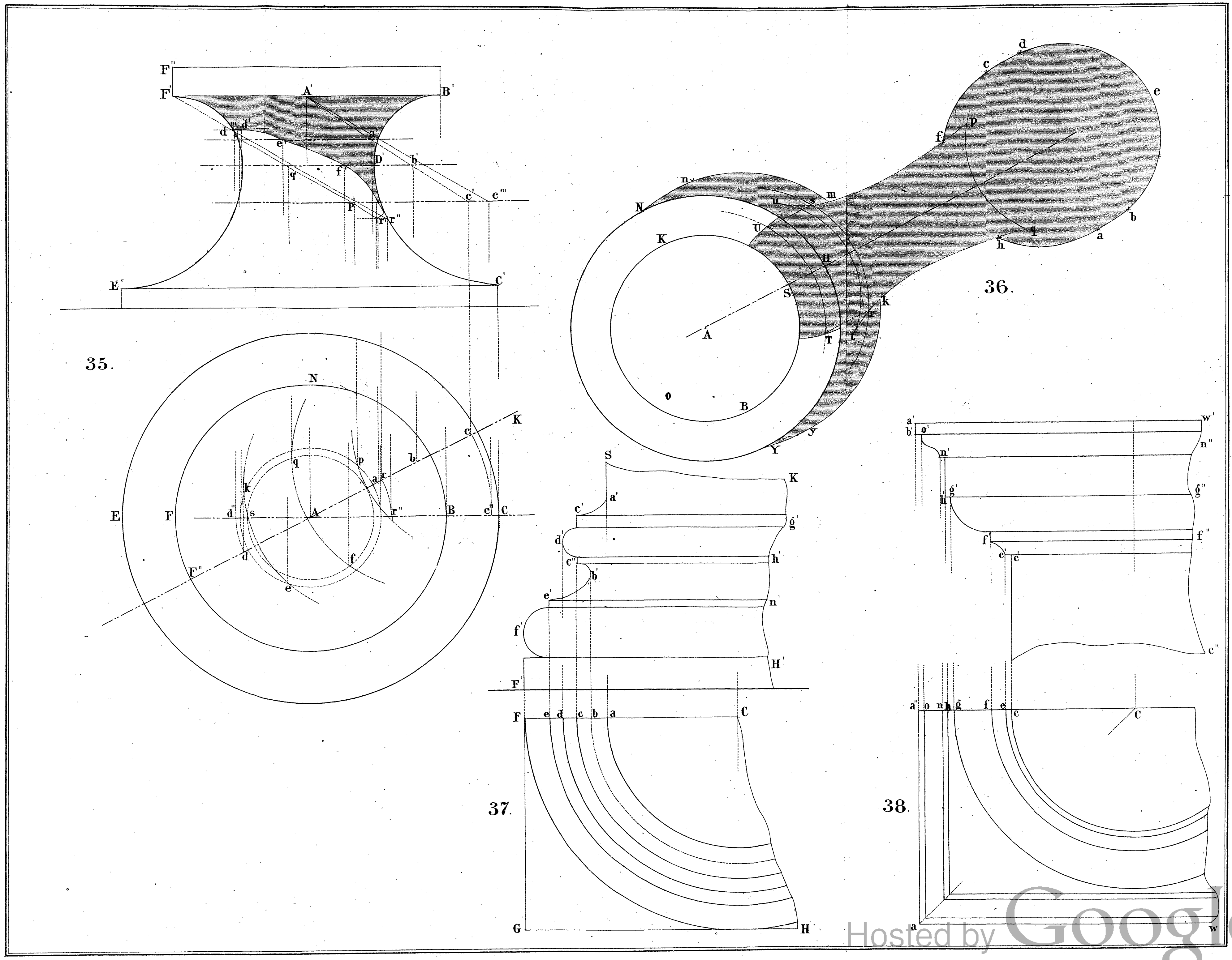
§ 1°.—*Geometrical Conditions for the adequate graphical representation of Form.*

95. The obviously essential geometrical feature of a surface is its continuity. But the bounding surface of a volume is represented, geometrically, as seen from a given point, by its *apparent contour;* which is only that line of the surface which is its visible boundary, as seen from that single point.

Geometrically, therefore, an infinite number of consecutive contours, seen from as many points of view, on each of the two opposite sides of a body, would be necessary to completely represent its continuity and conformation.

But the method of projections usually gives but two apparent contours; viz., those which constitute the two projections of a body. Furthermore, it would be graphically impossible to represent consecutive apparent contours; hence, purely geometrical diagrams, though enabling us to represent any required point of a surface, can never, of themselves, adequately represent the continuity of the inclosing surface of a body.

96. It is therefore our previous and definite conception of the surfaces to be represented, which enables the projections of



J. Bien, Lith. 24 Vesey St. N.Y.

surfaces practically to express the forms of those surfaces intelligibly. This may be illustrated by the difficulty usually experienced at first, in comprehending projections of new surfaces, as warped surfaces, before acquiring, from models or other sources, some idea of their form. On the other hand, the circular and rectangular projections of the familiar cylinder of revolution, perfectly represent it to the mind, because we so well know that *all* its right sections are equal circles, and *all* its meridian sections are equal rectangles.

97. By calling in physical considerations to the aid of geometrical ones, and confining ourselves to the sense of sight as a means of judging of the configuration of bodies, the action of light upon their surfaces would be observed at once. This leads to the following considerations:

§ 2°.—*Of the physical conditions for the visibility of Bodies, and an adequate representation of their Forms.*

98. The most immediately obvious result of the exposure of a body to the light, is the existence, upon the body, of a line of shade separating the illuminated from the shaded portion of the body; and the production of a shadow upon any adjacent surface from which light is excluded by the given body. The construction of the lines of shade and of shadow have, accordingly, claimed attention in all the previous problems.

But from (95) the curve of shade, alone, is insufficient, together with the apparent contour, to determine, unequivocally, the form of a body. This becomes further apparent, as follows: Having a given cylinder of rays, any curve traced upon its surface, may be the curve of contact of an infinite number of different bodies, all of which shall have the same projecting cylinder, and all of which will have this one assumed curve for their curve of shade. All these bodies, being inscribed tangentically in the same cylinder of rays, will, moreover, cast identical shadows on any other given surface.

99. See, at this point, Pl. XV., Fig. 55. NBEC is any opaque body. ABKD is a tangent cylinder of luminous rays, giving the *curve of shade* BHDG. Next, IFCJ represents a tangent projecting cylinder, or cylinder of visual rays reflected from the object, and gives the *apparent contour* FGCH. Hence the portion, E—FGCH, of the body, bounded by that contour, and the

portion, GDH, of its curve of shade, are visible. But besides the curve and that contour, there is only our imagination to suggest the real configuration of the visible portions of the surface on which no contour lines are shown. These portions may indeed have any sinuous configuration, though experience and association lead us to assume that the figure represents an ellipsoidal body, or something like one.

100. We therefore continue the examination of the action of light upon surfaces, in order to discover how their forms may be distinguished, and we find that a surface is illuminated in proportion to the directness with which the light falls upon it. See Pl. XV., Fig. 56. Here, let AB be the trace of a plane, tangent at T, to any curved surface. The space *ab* is illuminated by the beam of rays, M. The equal space *cd*, struck more obliquely by the light, is illuminated by the thinner beam N, while another equal space, *ef*, which is struck perpendicularly, receives all the light of the thickest beam, O. Hence, in this case, the curved surface is most highly illuminated at the point of tangency, T. Along the curve of shade, tangent planes coincide with the direction of the light, and therefore the given surface there receives no light. The curve of shade therefore is the darkest part of the surface.

101. Between the curve of shade and the brightest point, curves may be conceived lying on the surface, at all points of any one of which, tangent planes will make equal angles with the rays of light.

But these angles represent the angles made by the surface itself with the light along the supposed curve. Therefore the curves just supposed will be curves of equal illumination, and the illumination of the surface will vary in intensity from the maximum darkness at the curve of shade, where the surface makes an angle of 0° with the light, to the maximum brightness, where the surface is normal to the light.

Moreover, drawing *a*S, for example, the perpendicular width of the beam of light, it represents the sine of the angle S*ba*, made by the light with the tangent plane AB. Hence, in this view, the intensity with which a surface is illuminated is directly as the sine of the angle made by the light with that surface.

102. But again: Bodies are not seen directly by the light *thrown upon them*, but by such portion of that light as is *returned from them*.

Hence, in general terms, *the comparative amounts of light remitted to the eye, from the different points of a body, are, with its apparent contour, the means of judging of its form.*

103. Now, optical researches show, that, when light falls upon a body, a part is *extinguished*, or destroyed (absorbed according to the material theory), a part is *reflected*, and a third portion is *polarized, by refraction among the molecules of the surface.*

104. Strictly speaking, a body is visible, only as it shows its own proper color, which it does by means of the *refracted* rays which it remits.

Mirror-like surfaces *reflect* the rays just as they are received, and present images of all objects, from which light proceeds to fall upon them. They thus *indicate their presence* by their effects, but are strictly invisible, except that, in practice, there are no absolutely perfect mirrors; hence they are faintly visible by the few refracted rays coming from the molecules of their surfaces.

105. The law of *reflection* is, that the incident and reflected rays, at the same point of a surface, make equal angles with that surface.

Hence, as there will, taking the general case of a double-curved surface, be but one such point, for a given fixed position of the luminous point, and point of sight, *a perfectly polished double-curved body, exposed to a single source of light, would have but a single visible* point; and that, the one at which a normal line, or a tangent line or plane to the surface, would make equal angles with the incident ray, and the ray reflected to the eye.

The condition for the complete visibility of a perfectly polished body, therefore, is, that it shall be exposed to light from all sources.

106. The majority of the objects which present themselves to our observation, as a stone or piece of wood, are dull, or partially polished bodies, that is to say, their exterior, without failing to present an appearance of continuity, is nevertheless, by their porosity, broken up into minute asperities and cavities, as shown, greatly magnified, in Pl. XV., Fig. 57. These irregularities, though separately invisible, from their minuteness as compared with the size of the body, yet, under the action of the light, produce an aggregate effect, which is appreciable, since they are of considerable magnitude, as compared with the extreme tenuity of the molecules of light. Each asperity, therefore, is considered

as having a number of indefinitely small facets, one or more of which is in a position to return light to the eye.

But the exterior facets, lying in the perfectly continuous imaginary surface of the body, are larger than interior ones, which are withdrawn from polishing agencies; and are nearer each other, as seen in projection, in the vicinity of the brilliant point, L, than in the more directly viewed portion of the continuous ideal surface containing them; and, besides, the properly disposed interior facets may often, in obliquely viewed regions, be hidden by asperities in front of them, so that *the point on a dull body, which corresponds to the only visible point of a perfectly polished body, is the brightest point of that dull body.*

107. Taking into the account the secondary remissions to the eye, from facets which are illuminated by reflection from other facets, it appears that the whole illuminated part of *a dull body is rendered visible by a single source of light.*

Its shade, however, can become visible only by sending to the eye the light received from a secondary source, as from the atmosphere, or from surrounding bodies.

Hence, if Pl. XV., Fig. 55, represents a dull body, illuminated by a single source of light, all that part, E—FGDH, of the illuminated part, which is in the field of vision, will be actually visible. The completely unilluminated shade, GCDH, though in the range of vision, will be invisible, since it remits no rays to the eye.

108. But the theory of minute *reflecting* facets alone, is insufficient, in not accounting for the colors of bodies; since light that is merely reflected, retains the color which it had when coming in incident rays. It has therefore been presumed, and is confirmed by experiments, that bodies are visible, mainly in consequence of light remitted by them after polarization by refraction, due to vibrations in their superficial molecules. These vibrations are supposed to be in unison—so to speak—with those of light of the color presented by those molecules; and they make those molecules act as self-luminous points while illuminated.

109. Hence, besides the contours furnished by their projections, the conditions for the full representation of the continuity and forms of bodies, are, that their surfaces, being dull or porous, shall be illuminated from a primary and strong, and a lesser or reflected light; that their numberless facets shall be so distributed as to reflect increasing quantities of light, in successive rings of

equal reflection, from the curve of shade to the brilliant point, analogous to the curves of equal illumination (101) and that the vibrations of their superficial molecules, under the action of light, shall cause them to act, while exposed to the light, as self-luminous bodies.

Finally: *a true representation* of these apparent varying intensities of light and shade, exhibits graphically to the eye the continuity and the consecutive changes of form of surfaces, hence, in connection with the apparent contours of bodies, it adequately represents them to the eye.

§ 3.—*Of Brilliant Points and Lines.*

110. Most conspicuous and important, next to the line of shade of a body, is its *brilliant point*, or the element of *greatest apparent illumination* in its illuminated part. As these brilliant points can, moreover, usually be constructed without difficulty, they will now be more fully considered.

In treating of brilliant points thus in detail, the relative positions of the luminous point, and the point of sight, with respect to a given body, are first to be noted. Since either of these points may be either at a finite or an infinite distance from the given body, the four following combinations may arise:

Distance of the	
Luminous point.	*Point of sight.*
Infinite.	Finite.
Finite.	"
Infinite.	Infinite.
Finite.	"

111. The two former cases are properly treated under the head of "*scenographic projections*" or natural perspective, since the point of sight is there supposed to be at a finite distance from the object viewed.

The two latter cases may here be discussed, since, whatever the distance of the source of light, the point of sight is at an infinite distance from the object, which is characteristic of orthographic projections.

112. In speaking now of the position of the brilliant point upon any surface, the distinction between the *real*, and the

apparent brilliant point must be observed. The *real* brilliant point on a given surface, is the point at which the tangent plane to that surface is perpendicular to the direction of the light, for this point is in a position to receive the greatest amount of light per unit of surface.

Hence, if T, Pl. XV., Fig. 56, be the point of contact of a surface with a tangent plane, AB, to which the light is normal, T will be the *real* brilliant point of the given surface.

113. The *apparent* brilliant point of a surface, is that point at which a normal to the surface bisects the angle between the incident and the reflected rays at the same point (105). The apparent brilliant point is the only one which it is necessary to consider, in treating of the finished execution of shading.

114. On developable single curved surfaces, and with the luminous point and point of sight both at an infinite distance, the brilliant point expands into a brilliant line, whose location will be explained in detail, in connection with the problems of the next section.

SECTION II.

The Construction of Brilliant Points and Lines.

§ 1.—*Brilliant elements of Planes.*

115. *A plane illuminated by parallel rays* would be equally light in every part, and, neglecting the effect of the different depths of atmosphere, through which its different parts would be seen were it viewed obliquely, every part of it would *appear* equally bright.

This result would follow from the theory of the constitution of surfaces, explained in connection with Pl. XV., Fig. 57, for then a material plane surface of uniform texture would present a uniform distribution of facets, so disposed as to reflect rays to the eye.

A plane, illuminated by diverging rays, will have a brilliant point.

Problem XLII.

To find the brilliant point of a plane which receives light from a near luminous point.

Pl. XV., Fig. 58. Let the plane be vertical, and let TP be its horizontal trace; and let S be the luminous point. Also let TR be the direction of the reflected rays, which are parallel (111).

At any point, T, construct a line, TN, perpendicular to the given plane, and make the angle KTN=RTN. Then, through the luminous point, S, draw a ray, SB, parallel to KT, and B, its intersection with the plane TP, will evidently be the point at which the incident and reflected rays make equal angles with the normal to the plane. Hence B is the brilliant point required.

Remarks.—a. Observe, that as NT is normal to the given plane, the plane of the incident and reflected rays is perpendicular to the given plane. Hence, in this case, the ray SB is horizontal in space.

b. The construction of the vertical projection of the figure, and the solution of the problem when the given plane is oblique to the planes of projection, may now be left to the student.

§ 2.—*Brilliant elements on Developable Surfaces.*

Passing to *single curved surfaces*, we shall first consider *developable surfaces*—and these, at first, as illuminated by parallel rays.

Problem XLIII.

To find the brilliant element on a cylinder, illuminated by parallel rays.

First Solution.—Pl. XV., Fig. 59. Let ANB—A′B′ be a vertical right cylinder, and RO—R′O′ the projections of a ray of light. OE represents the direction of those reflected rays which reach the eye. Now revolve the plane, R′O′E, of the incident and reflected rays, RO—R′O′ and O′—OE, about its horizontal trace, O′E, into the horizontal plane of projection. The incident ray will then appear at R″O′—R‴O, and L‴O is

the revolved position of the bisecting line of the angle ROE (105). Then, by making the counter revolution, this bisecting line will appear at LO—L′O′, since it is in a plane R′O′E which is perpendicular to the vertical plane of projection.

Now we may suppose that a row of asperities is ranged along the element N—N′N″ in which the bisecting line pierces the cylinder. If then each of these asperities has one or more minute facets which are perpendicular to LO—L′O′, they will collectively reflect the rays contained in a vertical plane of rays through RO—R′O′, and thus form a brilliant line N—N′N″.

Second Solution.—In this solution, some of the facets of the little asperities are supposed to be arranged in parallel elliptical bands, indefinitely narrow, and contained in the successive parallel planes of incident and reflected rays.

See Pl. XV., Fig. 60, where the light is taken in the same direction as in the last figure, for the sake of easy comparison of the two. Those features of the construction, which are the same in both figures, are here omitted.

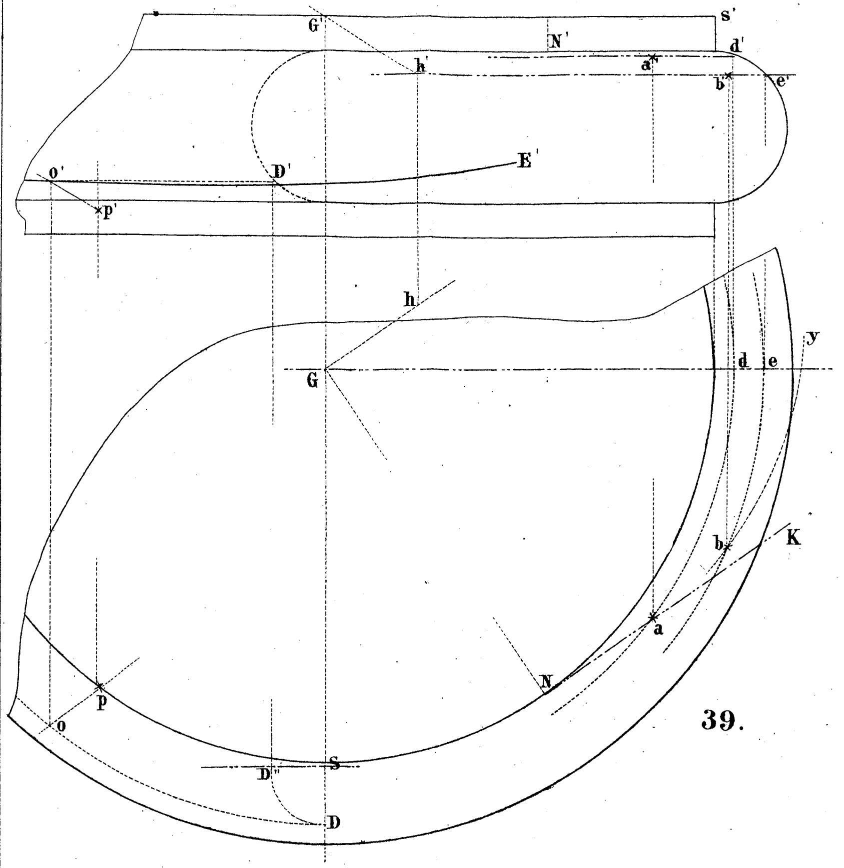
RO is the revolved position of an incident ray, OE is a reflected ray, and DO is the bisecting line of their included angle. A′B′ is the vertical trace of a plane of rays, perpendicular to the vertical plane of projection, and determining the narrow elliptical band A′B′, along which facets are supposed to be disposed so as to reflect light to the eye. A″CB″S is the revolved position of this elliptical band. O*n*, the bisecting line of parallel chords, as C*e* and *ab*, which are perpendicular to DO, determines N‴, the point at which a normal, parallel to DO, can be drawn.

In the counter revolution, N‴ returns to N; hence, if the convex surface of the cylinder be supposed to be formed of consecutive bands of reflecting facets parallel to A′B, a vertical row of them will be found on an element N—N′N″.

Remarks.—*a.* It should be remembered, that the plane of the incident and reflected rays is normal to the reflecting surface.

Otherwise, a ray striking a plane, at a point A, for example, might be reflected in a cone of reflected rays, generated by the revolution of the incident ray about a perpendicular to the plane at the point A.

But in the case of an absolutely unporous and polished cylinder, or cone, which is struck obliquely by the light, a plane of incident and reflected rays, situated as in the present example, cannot be made normal to the convex surface at N. Hence such a



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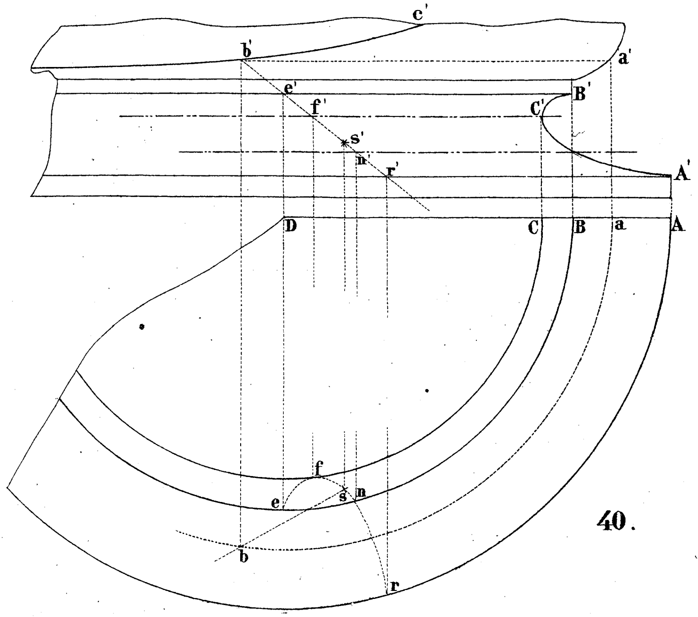

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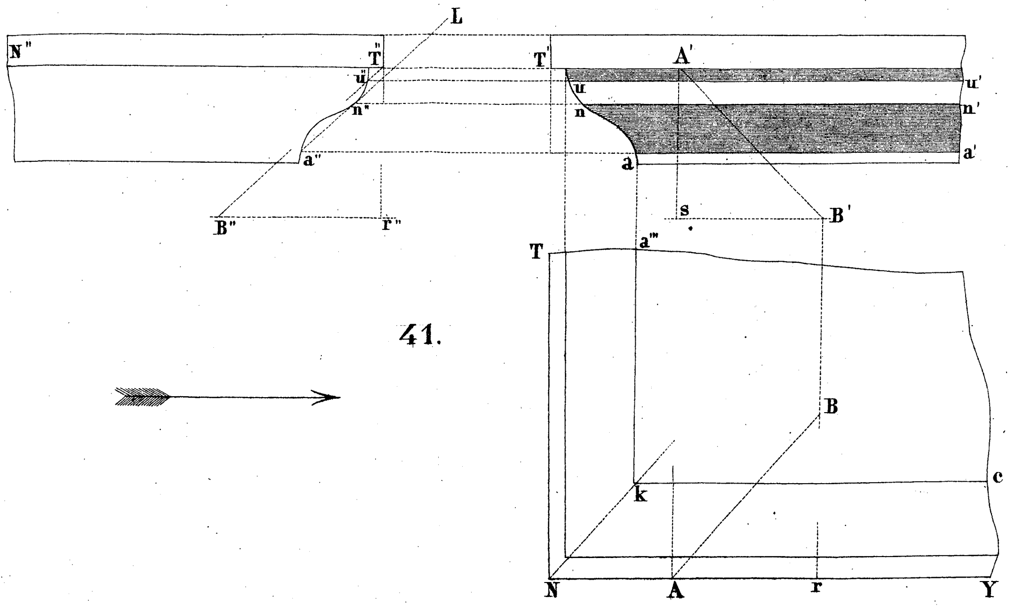

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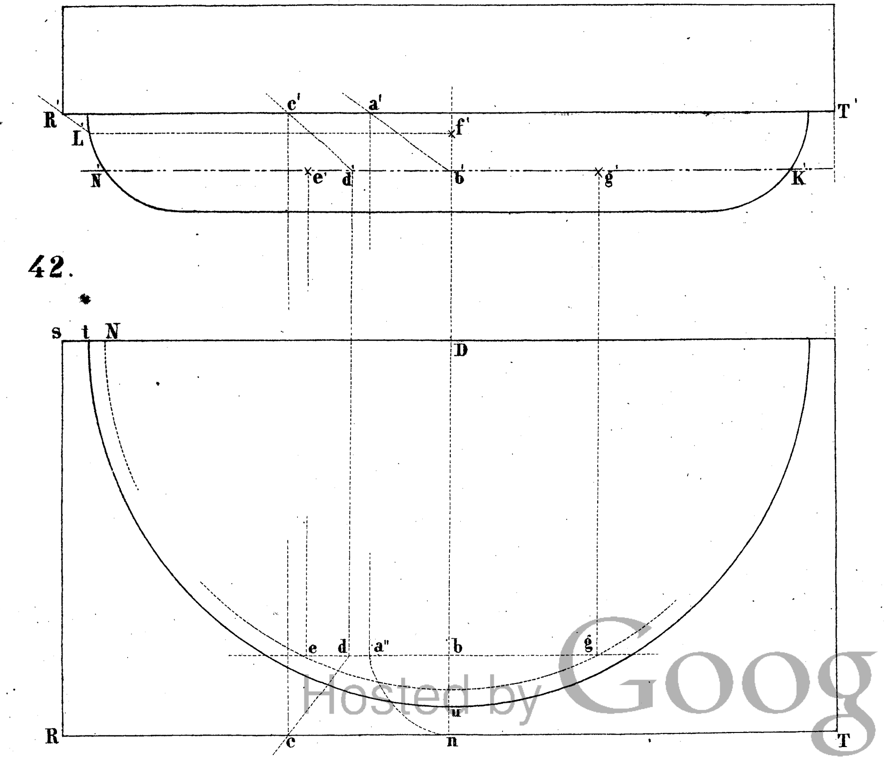

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cylinder, when illuminated from a single source of light, should be absolutely invisible throughout.

The existence of asperities of surface, presenting reflecting facets, is, therefore, a necessary condition for the visibility of the surface thus illuminated. Hence reasoning or experiment must decide which of the theories of the location of the reflecting facets, given in the two solutions of this problem, is true, since these solutions give quite different locations to the brilliant line.

The first construction is, I believe, the only one hitherto used, excepting the approximate one given in my "Elementary Projection Drawing," and it agrees with the formation of the artificial grain of the surface by the operation of turning.

b. Either of the solutions of this problem may be applied, by the student, in finding the brilliant element of a cone.

Problem XLIV.

To find the brilliant point of a cylinder which is illuminated by diverging rays.

Pl. XV., Fig. 61. Let L be the luminous point, and BDK the horizontal projection of the cylinder. The reflected rays being parallel, because the eye is at an infinite distance from the cylinder, let LG be a line in the direction of the reflected rays.

All lines through O, and in a plane perpendicular to the axis of the cylinder, will be normal to its surface. The cylinder being vertical, in the present case, this plane will be horizontal, and EO, FO, GO, etc., are normals to the cylinder.

Now, making $La=LE$; $Lb=LF$, etc., the curve abc is the line at each of whose points incident rays, as La, and reflected rays from a, etc., parallel to LG, make equal angles with the normals, as EaO, etc., through the same points.

Hence B, where this curve meets the convex surface of the cylinder, is the point of that surface at which the incident and reflected rays, LB and BY, make equal angles with the normal BO. That is, B is the horizontal projection of the required brilliant point. Its vertical projection is not shown, it being found by merely projecting B into the circle cut from the cylinder by a horizontal plane through the luminous point.

Problem XLV.

To find the brilliant point on a cone which is illuminated from a near luminous point.

Pl. XV., Fig. 62. In the last problem, the curve in which the normals pierced the cylinder, coincided, in horizontal projection, with the projection BDK of the cylinder, and was, therefore, not constructed. The cone, not having a vertical surface, this curve of intersection of the normals would appear as a distinct curve, which must be constructed.

Let VDTH—V′D′H be a cone of revolution, having its axis vertical; and let S, A′ be the luminous point.

Draw SA*b*″, in the direction of the reflected rays. At S, any line will make equal angles with S*b*″, and some incident ray, hence S is a point of the trial curve, containing the intersection of normals with incident rays. A is another point of the same curve, since SA, regarded first as an incident, and then as a reflected ray, makes the same angle with a normal VA at right angles to it.

To find other points of the trial curve SA*h*. At any point, as *b*″, on the line S*b*″, draw the horizontal projection, *b*″V, of a normal. Revolve this normal to the position *b*V, parallel to the vertical plane of projection. N′*b*′, perpendicular to the element V′D′—taken as the revolved position of the element V*v*—will then be its vertical projection, and A′N′ is the vertical projection (not drawn) of its primitive position. Now revolve this normal and the line S*b*″, about an axis perpendicular to the vertical plane at N′, and into the horizontal plane N′A″. It will then appear at A″N′—BV, and S*b*″ will appear at A″—S‴B. Now make S‴*h*″—not drawn—equal to S‴B, and *h*″ will be the revolved position of another point of the trial curve. In the counter-revolution, *h*″ returns in a short arc, *h*″*h*, parallel to B*b*″, and *h*, on the primitive projection, *b*″V, of the normal, is another point of the curve SA*h*.

Having now found one point of this curve, in the manner just described, others are easily found as follows. Revolve assumed points, as *p* and *s*, to *p*″ and *k*, and join *p*″ and *k* with V. Then arcs, with radii S‴*p*″ and S‴*k*, will locate points on *p*″V and *k*V—not shown—at distances from S‴, equal to S‴*p*″ and S‴*k*.

Then, by counter-revolution, as before, find c and y. The trial curve SAch can now be drawn.

It is necessary, in the next place, to construct the curve in which all normals, which intersect Sb'', pierce the conic surface. Take, for example, the normal Vb''. After revolution into the meridian plane AV, parallel to the vertical plane, this normal appears at Vb—$b'r'''$, normal to the revolved position, V′D′, of the element Vr, which is in the meridian plane through Vb''. Hence r''',r'' is the revolved, and r,r' the primitive position of the intersection of this normal with the conic surface.

Other similar points being found in like manner, the curve uer—$e'n'r'$ may be sketched as the locus of the intersections of all normals, through points of A′—Sb'', with the conic surface. This curve intersects SAh, the locus of the intersections of normals with incident rays, at n,n'. This point is therefore that point, on the conic surface, at which the incident and reflected rays, Sn and nE, make equal angles with the normal, nV, at that point. Note, however, that these angles are not equal in projection, since their plane is oblique.

Remark.—We may construct the point in which the normal at any point will pierce the line Sb''. Thus, let F be the revolved position of any point at which a normal is to be drawn. FK, perpendicular to V′H′, is the revolved position of this normal. The revolution of FK, about V′K as an axis, will generate a cone, whose surface will be normal to that of the given cone; then the intersection of this auxiliary cone with the line Sb''—A′, will give the point, which, when joined with V,K, will be the normal required.

§ 3.—*Brilliant Points on Warped Surfaces.*

Problem XLVI.

To find the brilliant point of a warped surface, when illuminated by parallel rays.

The solution of this problem involves the construction of a normal, parallel to a given line, L, viz., to the bisecting line of the angle included by an incident and a reflected ray.

But since these rays make equal angles with a tangent plane

which is perpendicular to the supposed normal, the construction of the normal may be effected by the construction of a tangent plane, perpendicular to a given line. But, again, since all planes which are perpendicular to the same straight line, are parallel, the operation, just proposed, is equivalent to the construction of a *tangent plane parallel to a given plane.*

To construct the tangent plane, draw any two lines, A and B, not parallel, but each perpendicular to the given line, L. A system of tangent lines, parallel to A, will constitute a cylinder, tangent to the warped surface. Another such system, parallel to B, will form another tangent cylinder. The intersection of the curves of contact of these cylinders will be the point of tangency, on the warped surface, of a tangent plane perpendicular to L, and therefore the point of intersection, I, of the required normal, parallel to L. Hence I will be the required brilliant point.

Problem XLVII.

To construct the brilliant point on a screw, when illuminated by parallel rays.

Produce, each way, several elements on each side of the meridian plane parallel to the vertical plane, the axis of the screw being vertical, so as to draw, tangent to them, a sufficient arc of that meridian curve of the screw, which lies in this meridian plane. Then, as in Prob. XLI., find the bisecting line, N, of the angle included between an incident and a reflected ray. Revolve the line, N, till parallel to the vertical plane of projection. Draw a line, parallel to this revolved position, and normal to the meridian curve, just named, at a point, *n*. This point, *n*, will be the revolved position of the brilliant point. In counter-revolution, it will return *in a helical arc*, and its projections will appear on the projections of N.

This construction can be wrought out by the student.

§ 4.—*Brilliant Points on Double-Curved Surfaces.*

Problem XLVIII.

To find the brilliant point on any double-curved surface, when illuminated by diverging rays.

Let Pl. XV., Fig. 63, represent any double-curved surface, not of revolution. Let S be the luminous point, and SR′ a line parallel to the reflected rays. Let $n'Nn''$ be the curve in which all normals, as $R'n'$, RN, and $R''n''$, intersect the given surface. Also, let N′NN″ be the locus of intersections of these normals with incident rays from S. That is, SN′=SR′, SN=SR, and SN″=SR″, etc., so that if the rays, SN′, etc., were drawn, the angles at N′, etc., made by the incident ray SN′, etc., and normals R′N′, etc., would be equal to the angles at R′, etc., made by the reflected ray SR′ with the same normals. Then N, the intersection of the two curves just described, is that point on the given surface, where the incident and reflected rays make equal angles with the normal RN. Hence N is the brilliant point.

The practical difficulty, in case of the class of surfaces considered in this article, and also in case of warped surfaces, is, that, unless they are surfaces of revolution, there is no convenient direct construction of the required normals. Hence the remaining problems embrace only the construction of the brilliant points of double-curved surfaces of revolution, when illuminated by parallel rays.

When such surfaces are exposed to diverging rays, their brilliant points are found as in Prob. XLIV.

Problem XLIX.

To find the brilliant point on a sphere, illuminated by parallel rays.

Pl. IX., Fig. 26. A vertical plane, through the centre of the sphere, is here regarded as the vertical plane of projection. OL is the vertical projection of a ray through the centre of the sphere. As no horizontal projection of the sphere is shown, O*r* may be assumed as the ray, after being revolved into the vertical plane

of projection, about LK, a line of that plane, as an axis. Then, also, OP will represent the direction of the reflected rays. Hence N′O, the revolved position of the bisecting normal of the angle *r*OP, gives N′ as the revolved position of the brilliant point. By a counter-revolution about LK, its primitive position, N, is found.

Problem L.

To find the brilliant point on a piedouche.

Pl. X., Fig. 31. A*i*—S′*i*′ is any incident ray intersecting the axis A—T′A′. Let S′—AW represent the direction of the reflected rays, as the piedouche is seen in vertical projection. Revolving the assumed ray around AW, as an axis, till it is parallel to the horizontal plane of projection, it appears at S′*i*‴—A*i*″. Then draw A*q*, the bisecting line of the angle *i*″AW, and draw an auxiliary line, *i*″W. In the counter-revolution, *i*″,*i*‴ returns to *i*,*i*′, the line *i*″W returns to *i*W, and *q* to *p*, whence it is projected to *p*′, because the plane WA*i* is perpendicular to the vertical plane of projection. Now *p*A and *p*′S′ are the projections of the bisecting line of the angle *i*AW. The brilliant point is the point of contact of a tangent plane which is perpendicular to this bisecting line. Then revolve A*p*—S′*p*′, till parallel to the vertical plane, taking A—T′A′ for an axis, when it will appear at A*p*″—S′*p*‴. Next, draw the normal *jl*, parallel to *p*‴S′, and *l* will be the revolved position of the brilliant point. Construct the projections of the horizontal circle through this point, and V,V′, its intersection with the meridian plane through the bisecting line *p*A—*p*′A′, is the primitive position of the required brilliant point.

Remark. The student may now construct the brilliant point on any of the double curved surfaces of revolution shown in the preceding plates, or upon a warped hyperboloid of revolution.

CHAPTER II.

OF THE REPRESENTATION OF THE GRADATIONS OF LIGHT AND SHADE.

116. In observing a shade, or a shadow, two things arrest the attention, its *position*, and its *intensity*.

To find the *position* of shades and of shadows upon surfaces, was the object sought in "Book I." Now, we are to determine their varying *intensities*, as affected by the several circumstances presently to be enumerated. Furthermore, the varying apparent intensity of the illumination of those parts of surfaces which are exposed to the light, will be affected by the same circumstances. Hence the gradations of the lights, and of the shades and shadows of surfaces, will be discussed together.

117. The particulars affecting apparent intensities, and the gradations of lights and shades, and which will next be discussed in detail, are arranged under the following heads:

1°.—Effects due to the laws of illumination and vision.

2°.—Those due to the supposed infinite distances of objects from the eye.

3°.—Those due to secondary sources of light; including the air as an illuminating medium.

4°.—Those due to the nature of different surfaces.

5°.—Those due to the atmosphere as an absorbing medium.

6°.—Those due to variations in the intensity of the light.

7°.—Those due to the forms of bodies.

8°.—Those due (on the drawing) to drawing materials and processes of manipulation.

118. *Effects due to the laws of illumination and vision.*

a. *Law of illumination.* The intensity of light, or of the degree of actual illumination of the same surface, at different distances from the source of light, varies inversely as the square of the distance from the luminous source.

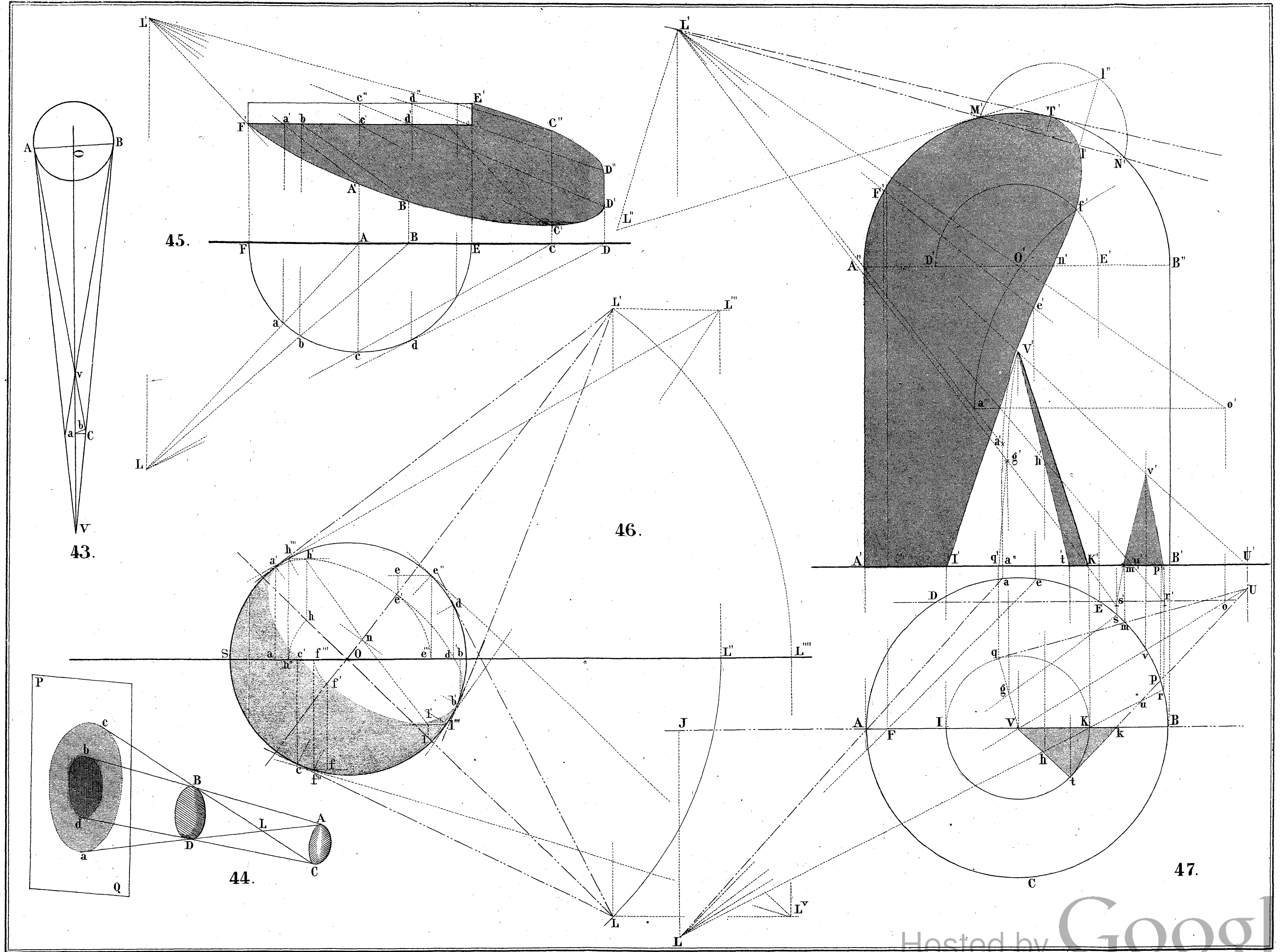
b. Case where the light comes from an infinite distance. The principle just stated applies, sensibly, whenever the source of light is at a finite distance. But when the light proceeds from an infinite distance, terrestrial bodies are all at substantially the same distance from it, and when similarly situated with respect to it, may be considered as equally illuminated.

c. Law of apparent brilliancy. From remark *a* it follows, that if a surface be illuminated in a given degree, its apparent brilliancy should be inversely as its distance from the eye, but its image being reduced in the same proportion, the retina receives light at the same rate or amount per superficial unit of its surface. Hence, practically, the apparent brilliancy of a surface of given brightness, and in a given position, will be the same at whatever distance it is seen; if we disregard atmospheric effects.

d. Illustration. The last remark is easily illustrated. See Pl. XV., Fig. 64.

Let A be a visible point of an illuminated surface, and let PQ be the pupil of the eye. The point A remits to the eye the cone of rays APQ, whose base is the pupil PQ. These rays converge in the eye, forming, on the retina at R, the image of the point A. If now the object be removed to A′, double the distance AC, the base, PQ, remaining of constant size, the section of the new cone, at the distance equal to AC from its vertex, will evidently have an area equal to one-fourth of PQ. But sections at a constant distance from their vertices measure the relative angular capacities of the two cones. Hence the eye at *double the distance* AC receives *one-fourth* as much light from the point A as at the distance AC.

But to apply this result practically, we must consider more than a solitary point. Two points will be sufficient. Then let B be a second point on the body AB. The axis, B*r*, of its cone of rays gives *r* as the focus of those rays on the retina, and R*r* as the space on the retina, illuminated by rays from the space AB on the object. Now remove AB to double the distance AC, viz. to A′B′, and the point B′ will be imaged at *r*′, one-fourth as brightly as at *r*. But R*r*′ is half of R*r*, and hence the area, of which R*r*′ is the diameter, is one-fourth as large as that whose width is R*r*. While, therefore, A′B′ remits to the eye one-fourth of the light that AB remits, that light is received by the area, R*r*′, of the retina, which is one-fourth of the area R*r*. Hence the



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rate of illumination of the retina, for a unit of surface, is the same in both cases.

e. In discussing this nice point, on which learned authors differ, care must be taken not to confound aggregate *amount*, and consequent *dazzling effects* of brilliancy, with its *rate* or *intensity*. A square foot of sun-lightened snow is as intensely bright, as to its degree of illumination, as a whole snow-bank, but the *total amount* of light from it, and *extent of retina* affected by it, is less than in case of the whole snow drift. Hence the *amount* of dazzling effect is less. So a star, as brilliant as the sun, can easily be gazed upon, though its *sensibly equal intensity* of brightness with the sun is shown by its twinkling, which is its dazzling effect upon a mere point of the retina; an effect which has only to be repeated in the same degree on many points at once of the retina, in order to become blinding.

f. Shading of the penumbra. When a penumbra (85) is occasioned, as it is, by a near radiant body of sensible size, it receives light from a larger and larger portion of the luminous body, as it approaches its own outer boundary. Hence its shaded representation would be lighter and lighter in the same direction, unless prevented by the form of the surface containing it, or by other special circumstances.

119. *Effects due to the infinite distance of objects from the eye.*

a. Objects at an infinite distance seen as points. At the outset, and as a preliminary remark, it should be noted that supernaturally acute powers of vision are required to perceive an object at all, at an infinite distance. This may be proved by a reference to the laws of vision, as follows:

Each molecule of the surface of an illuminated object, having facets so disposed as to diffuse light in all directions, remits to the eye a cone of rays whose base is the pupil of the eye (118*d*). This cone of rays, in passing through the lenses of the eye, converges to a point in its axis, and on the retina, which is the image of the point on the object from which the rays proceeded. Each point of the object thus produces its image, and these image points, collectively, form the superficial image of the object.

But now conceive the object viewed to be at an infinite distance from the eye. Frusta, of finite length, of all the cones of rays remitted to the eye, and having the pupil for their common base, will sensibly coincide in a single surface, which will be

sensibly cylindrical. These frusta of rays, having thus a common axis, converge to a single common point on the retina, which is the image of the object. Now in order to be sensible of this image, consisting of but a single point, the retina must be of *extraordinary sensibility*—unless the light be of extreme intensity—also, its nervous texture must possess *absolute continuity*, in order to be impressible, strictly, at every point.

b. Why represented as having sensible magnitude.—In the two respects just named, then, the eye must possess extraordinary power, in order to perceive infinitely remote objects; but, since they should appear as points, why are they represented as of sensible magnitude? For the same reason that applies to ordinary vision. A giant nine feet high, and ninety feet distant, appears of the same size, nominally, or in a merely geometrical sense, as a boy three feet high, and thirty feet distant. But, practically, we say that each appears as we realize that it appears, and the appearance that we realize, *i. e.*, the one that *seems real* to us, is determined by our knowledge, derived from other sources, of the real sizes of objects. Knowledge, therefore, of the *actual sizes* of objects, may make us *believe* that an infinitely remote object, seen with the organs above described, does appear of sensible size, according to that knowledge; rather than as a point, according to its minute image on the retina.

c. Theory of exaggeration of effects in shaded projections. But again, eyes of such acute sensibility, would also indicate with corresponding vividness the modifications of light and shade due to the causes already stated (117), hence the usual practice of greatly exaggerating effects, is quite rational in shading objects when shown in projection.

This exaggeration appears principally in two particulars: *First*, in the extremely vivid contrast, between the element of shade and the brilliant point, which does not appear in common experience, but which appears natural, and agreeably suggestive of the forms of objects when represented in shaded projections. *Second*, in the effect allowed in representing surfaces at different distances. This is shown, *first*, in the manner of shading single surfaces seen obliquely; and, *second*, in the tinting of a series of parallel surfaces, at slightly increasing distances, with tints of increasing darkness.

120. *Effects due to secondary sources of light; including the air as an illuminating medium.*

a. Direction of the strongest secondary light. A surface will diffuse light in quantities proportioned to the amount received. It will receive most when struck perpendicularly. Therefore the greatest intensity of light from the particles of air, and of surrounding surfaces, will be in a direction opposite to that of the primitive rays.

b. Of the relative darkness of different portions of the shade on a given surface. From (*a*) it follows that the shade of a body will have its faint brilliant element, due to the secondary lights, just as the illuminated part has its own brilliant element, due to the primary light. Therefore the shade is shaded lighter, in receding from the line of shade, which is the darkest line of the body in reference to both lights.

c. Of the relative darkness of different parts of a shadow. Any point in a shadow receives diffused light from surrounding objects, except within a cone whose vertex is the point, and whose base is the curve of contact of the cone with the body casting the shadow. Therefore, generally, the further a point of shadow is from the object casting it, the more diffused light it receives, and the lighter it is.

The *general principle* just before named, must, however, be applied by inspection to each particular case of shadows on plane, convex, and concave surfaces. In case of a vertical prism casting its shadow on the horizontal plane, the shadow would be darkest at its junction with the base of the prism; a result which is confirmed by observation.

d. Optical, and actual effects of near surrounding objects. When an illuminated surface is placed quite near a shade, it actually illuminates a small neighboring region of that shade to a noticeable extent, and that region should therefore receive a lighter tint, in a drawing showing both bodies.

Besides this *actual effect*, there are *optical effects*, due merely to the mutual influence of contrasts, as when a shadow falls on an illuminated surface, and when two tints of widely different intensity are brought together. Here, the light surface appears lighter, and the dark surface darker, by contrast, as when black velvet laid upon black cloth makes the latter look lighter. Likewise, complementary colors heighten the effect of each other, when brought together. Thus, purple and gold, or blue and orange, each look more brilliant when side by side, than when seen separately. Such contrasts are presented in the shaded

drawing of an assemblage of objects, as well as by the original objects, hence the shading need not be exaggerated in order to produce the proper effect. Thus, if a shadow on a brilliant surface be of nearly uniform intensity, as it would be on a plane perpendicular to the light, its edges would, as they should do, *appear* darker than the interior portions, by contrast with the high light around them.

e. Of the relative darkness of a shade and a shadow. Some interesting principles are suggested by considering the relative darkness of a shade and of a shadow, under various circumstances. For example, let a shadow fall on the surface of a sphere. Any point in this *shadow*, will receive scattering rays of reflected and refracted light from all illuminated bodies and particles, in the hemisphere of space bounded by a tangent plane at the given point, and exterior to the cone, whose vertex is the same point, and whose base is its curve of contact with the body casting the shadow. A point in the *shade* of the sphere, near to which no other surface receives a shadow, will be illuminated by reflections from the entire hemisphere of space, bounded by the tangent plane at that point. Hence the latter point would be somewhat lighter than the former; as also from the fact of its receiving the strongest atmospheric reflection (*a*).

f. In the case of the line of shade, as compared with a shadow on the illuminated part of a body, the shadow is the darker of the two. For, as the atmospheric reflections are strongest in a direction opposite to the light, they are weakest or nothing in the same direction as the light. Hence shadows cast upon surfaces in the vicinity of their brilliant points are darker than the lines of shade of those surfaces. The shadow of the abacus upon the cylinder, for example, is darkest at the brilliant element of the cylinder, and fades away to the element of shade, where it disappears.

g. When the primary and secondary lights are of nearly equal intensity, as seen in the comparatively uniform diffusion of light in a cloudy day, the contrast between the brilliant point, and the curve of shade, and the shadows, would be diminished. The latter would appear lighter, and the former darker, than when exposed to a single strong light.

121. *Effects due to the nature of surfaces.*

a. The nature of surfaces may vary, by being dull or polished, or by being differently impressed, in their molecules, by different

colored rays. While no surfaces are of absolutely perfect polish, owing to their porosity, yet, as they approximate to such a polish, it may be possible to express that fact graphically by the manner of shading them.

A perfectly polished body, we have seen, has a single, well defined brilliant point. Hence, other things remaining the same, *the more polished the surface, the smaller may be the region of light shading containing the brilliant point;* and *the more dull the surface, the broader may be the area of light shading*—as if each large asperity on a dull body, which is pretty directly illuminated, had its own somewhat conspicuous brilliant point.

b. As already stated (104) bodies are, strictly speaking, not seen by their *reflected* light, but by that which they *refract.* Reflected light merely furnishes images of the objects which send rays to the reflecting surface, just as a surface echoes sound without imparting to it any new quality of tone derived from the echoing surface.

In accounting for the colors of bodies, an analogy is supposed to exist between them and musical instruments, or resonant volumes of air. Many persons may have observed that upon sounding various notes, in a clear and strong humming manner, in a closed closet or small room, some of them will, without extra effort, sound much louder than others, as if the whole body of air in the room were excited to musical vibrations in unison with the loud sounding note. So surfaces, when exposed to the pulsations of light, seem to respond, in the vibrations of their molecules, only to the vibrations of rays of a certain color. By thus responding, these surfaces become, for the time, luminous bodies, emitting, however, only such colored rays as their nature causes them to be excited by.

Now, as bodies, in proportion to the perfection of their polish, reflect more and more light, less will remain to express their color by the refracting process just explained. Hence the higher the polish, the less clearly will the proper color of a surface be revealed.

c. It may be added, as a matter of interest, to the illustration of the closed room, that such inclosed spaces are resonant, though in a less degree, to notes which differ but little from the principal note. So it is found, also, that the colors of bodies are not absolutely pure, but are always mixed with small portions of the neighboring colors of the spectrum. Thus the light from

a clear yellow flower, when dispersed by a prism, shows some orange and green rays, mixed with the pure yellow ones.

122. *Effects due to the atmosphere as an absorbing medium.*

a. Appearance of an illuminated surface seen obliquely. The atmosphere, not being perfectly transparent, extinguishes a portion of the light coming through it, and this portion is greater in proportion to the extent of air traversed. But we attribute to a surface the amount of light entering the eye in the direction from it to the eye. Hence the most remote portions of an illuminated surface are the darkest in appearance.

b. Appearance of a surface of shade seen obliquely. A surface of shade only remits to the eye tertiary rays, as they may be called, that is, rays received by reflection from surrounding objects. Hence the atmospheric reflections coming to the eye in the direction of the shaded object, may be supposed to be stronger than the light from the shaded object itself. But, as before, we attribute to the object the light coming to the eye in the direction of it, while the more remote it is, the greater is the body of intervening air which sends light from its own molecules to the eye. Hence the more remote a surface of shade is, the lighter it appears, or the nearer it appears to be of the same brilliancy of the atmosphere generally. This is very evident in nature, when we compare the lights and shades of distant buildings, or hills, with those of near ones.

c. Results of the two preceding principles. 1°. A series of illuminated parallel surfaces, seen perpendicularly, will appear darker and darker, as they become more distant. 2°. The reverse will happen, if these surfaces are in shade. 3°. Near shadows will appear intense, and remote ones faint. 4°. Portions of illuminated curved surfaces, which are more distant than the brilliant point, or line, will appear *darker* than that point or line. 5°. Portions of the shade of curved surfaces, which are more distant than the line of shade, will appear *lighter* than the line of shade. 6°. Curved surfaces will appear of a more uniform tint, the more distant they are, the lights being darkened, according to (*a*), and the shades dimmed, according to (*b*).

d. Limitation of the last principle. The last principle applies, however, to the relative depths of a body of air of *finite thickness*, through which bodies are seen from an *infinite distance*. See (119).

e. Effect of the air on apparent color. The *blue color* of the air,

due to the rays remitted from it *by refraction*, as explained already, is attributed to objects seen through it, as well as the *intensity* of the atmospheric rays. Hence distant objects appear of a bluish cast, which increases in clearness with their distance. It is thus that, in a colored drawing, a distant object may be distinguished from the dull shading due to uniformly diffused light (120*g*).

123. *Effects due to variations in the intensity of the light.* Interesting results follow an examination of the effect of lights, each received mainly from one direction, but varying in intensity. All so-called transparent media absorb some light, and, for the present purpose, it is sufficiently correct to assume that, for light passing through any given medium, and in greater quantity than can be absorbed by the medium, the absolute amount absorbed will be equal for all degrees of intensity. It is important here to observe that this absorbed light is, as it were, latent, and does not at all render the absorbing body visible.

It follows that in case of a feeble light, where the total amount absorbed by the air and by surrounding objects is large in comparison to the whole amount of light emitted, the diffused light caused by reflections and refractions among the particles of air, and the light remitted from surrounding objects, will both be small, but very little light will be thrown upon the shade, or into the indefinite shadow in space of a body, under these circumstances, and hence this shade and shadow will approach in darkness to those formed under the supposition that they received no light, and were absolutely black (105).

Moreover, the diffused light spoken of, partially illumines objects within the indefinite shadow; but if it be feeble, it will all be absorbed by those objects, and they will therefore, as shown above, not throw any light into the indefinite shadow, or upon the shade of the body. For this additional reason, therefore, will the shades and shadows due to a feeble light be intensely dark, as compared with the illumined part of the body. This conclusion corresponds to the great blackness of the shades and shadows observable on and about buildings by moonlight. To return, now, to the case of clear sunlight, and then to apply these principles, it appears that, while the absolute amount of absorption remains the same, there is a large excess of light, available for producing diffused and reflected light, of sufficient brightness to produce reflection from bodies in the shadow upon the shade of an opaque body, so that enough light will reach

this shadow and shade, to mitigate their blackness considerably.

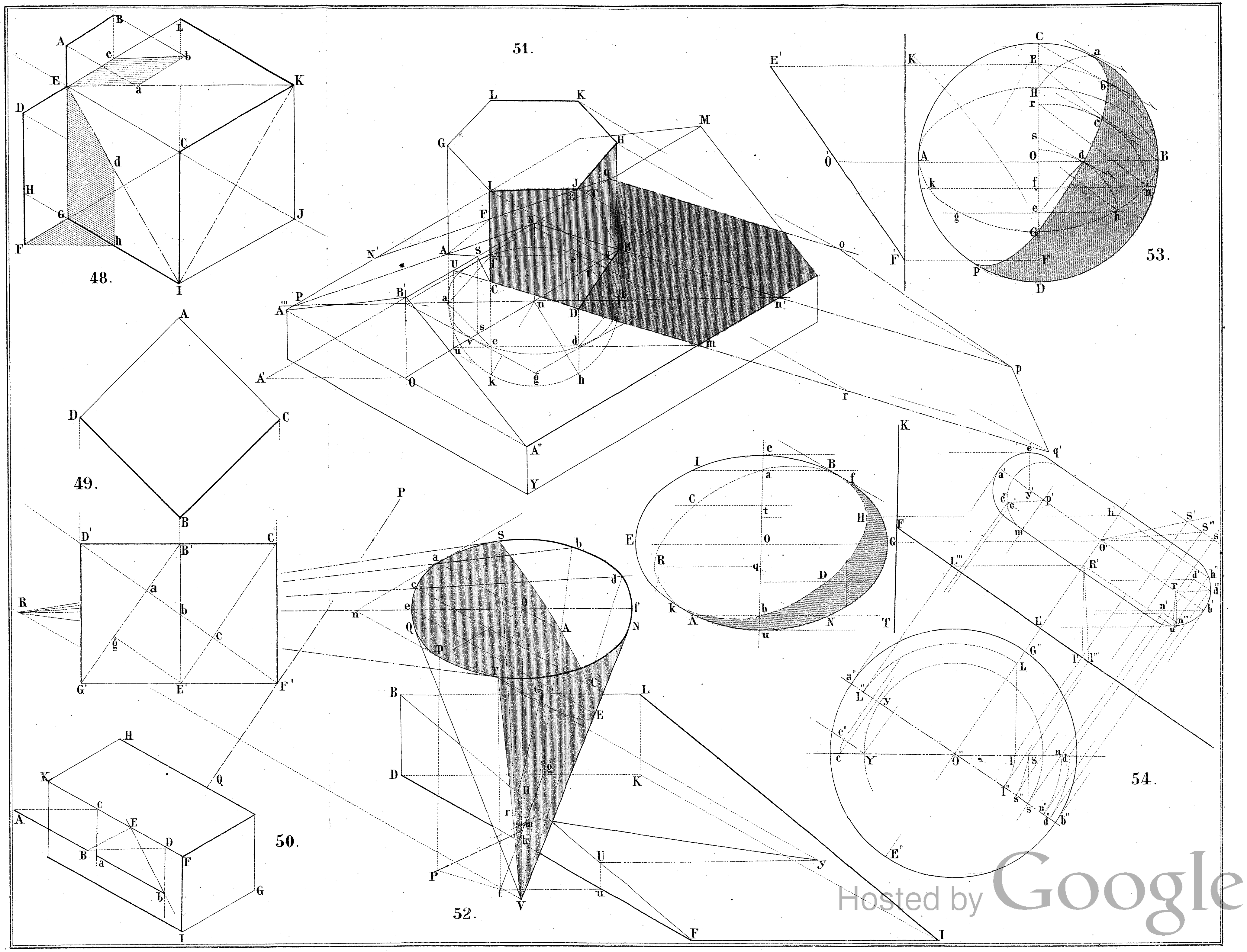
124. *Effects due to the forms of bodies.*

a. Form of the curves of equal shade. The lines of equal tint on different bodies, will be either similar or dissimilar. They will be similar on developable surfaces, where they are simply rectilinear elements, and on spheres, ellipsoids, etc., where they will generally be circles, ellipses, etc. But they will be dissimilar on the more complex surfaces, such as the torus and piedouche. In the latter case, they will be gradually transformed, from a ring surrounding the brilliant point, to curves more and more nearly similar to the curve of shade. This should be remembered in distributing the tints upon such surfaces.

b. Distribution of points of equal shade. Points of equal apparent brilliancy are not, however, symmetrically placed around the brilliant point as a centre. That is, on a sphere, for example, the curves of equal illumination are not circles, having their centres in the diameter through the brilliant point. See Pl. XV., Fig. 65, which represents a sphere, with the plane of the paper taken as a plane of incident and reflected rays, as LO and OR′. Then B, the middle point of the arc AC, is the brilliant point.

Now from (120*a*) the small superficial element, shown as having A for its centre, is the element having the greatest *actual* illumination. This element returns the broad beam of rays, LA, partly by reflection (106) in the condensed beam AR. An equal element, equidistant from the brilliant point, as at C, receives only the narrow beam of rays L′C, which, furthermore, it remits in the diffuse beam CR′. It is now quite evident, from a comparison of the breadth of the two reflected beams, that, if they diffused equal amounts of received light, the intensity of this diffused light, and consequently the apparent brilliancy of the elements A and C, would be *inversely as the sine of the angle made by the reflected ray with a tangent plane at the point from which it proceeds.*

But not only is the beam CR′ more diluted, so to speak, than AR, so far as the space filled by a given amount of light is concerned, but it actually contains fewer rays, viz., those of the narrow incident beam L′C. Hence, much more, is the element at A, brighter than the element C, which is at the same distance from the brilliant point, B.



J. Bien, Lith. 24 Vesey St. N.Y.

In fact, the superior brilliancy of B over all other points, can be fully accounted for, in this view, only by taking account of the effect of distance (122*a*), and by supposing that some of the reflecting facets of molecules in the element A, which is seen obliquely, are hidden by others in advance of them, so that all the rays of the beam LA are not visibly reflected.

Therefore, finally, from the above, we conclude, that on cylinders, cones, spheres, etc., points which are equidistant from the brilliant line, or point, should not receive equal tints; and observing, in Fig. 63, that tangent rays parallel to LA determine the curve of shade, that the darkest of two such points should be the one which is between the brilliant element and the visible projection of the element, or curve, of shade.

c. Relative illumination of apparent contours. It may be separately noted that, as a sort of balancing between the results of the light received by a surface, the light reflected and refracted from it, and the obstruction of some of these remissions, there will be a narrow space along the apparent contour of curved surfaces, which will appear lighter than the portions a trifle further from that contour; for this space, which is very narrow in projection, is wide in reality, and so receives a broad beam of light; which, however, it reflects within a narrow compass.

d. Rate of variation of shades on different surfaces. A consideration of the forms of bodies shows that there must be variations in the rate of gradation from light to dark, on different surfaces. For example, see Pl. XV., Figs. 59 and 60, equal vertical strips of shade on the cylindrical surface will not appear equal in vertical projection, and the shading, used to express the curvature of its convex surface, would therefore proceed rapidly from dark to medium shades, and then, more gradually, from medium to light shades. But on the vertical plane seen obliquely, Fig. 58, similar equal strips would appear equal in projection, and the shading used to express the different distances of these strips from the eye (122), would proceed at a perfectly uniform rate from dark to light.

e. The proper shading of bounding edges. The theory of the form of bounding edges, should be observed in the execution of shading. These edges are supposed not to be mathematical lines, but to be rounded, through the ultimate imperfections of workmanship, and the attrition of flying particles. Thus, an edge of a prism is to be regarded as minutely cylindrical; and

the circumference of the base of a cylinder is a portion of a torus, whose meridian section is a very minute quadrant.

These edges, even when in shade, will have their own brilliant points, due to the scattering light which they may receive from surrounding objects, and to their superior smoothness, consequent on greater exposure to polishing agencies. Accordingly, very pleasing effects are produced, especially in an assemblage of shaded figures, as in a machine drawing, by shading the edges of each form, according to the principles here stated.

f. Illustrations. First. On a shaded drawing of a vertical cylinder of revolution, the upper edge should everywhere be lighter than the bordering portions of the convex surface. The lower edge of its illuminated portion would be darker than the convex surface, but the lower edge of its shade should be lighter than the contiguous shade, though rather darker than the upper edge of the shade.

Second. The successive parallel plane surfaces, forming the front of an edifice, are distinctly shown in projection, by thus treating their edges, which are mainly horizontal and vertical quarter cylinders. Upper and left hand edges, when exposed to the light, are left blank, or of a light shade, for a very narrow space. Lower and right hand edges, are quarter cylinders containing elements of shade, and may be ruled with a tint a little darker than that of the flat surface bounded by them.

The light edges may be ruled with white, if not left blank in tinting. This course would produce the most striking effect in drawings on tinted paper. By thus distinguishing parallel surfaces by the treatment of their edges, we avoid the necessity of an undue exaggeration of the difference between their shades, as explained in (122 *c*).

125. *Effects* (on the drawing) *due to drawing materials and processes of manipulation.*

a. Since the amount of remitted light received by the eye, both white, or reflected, and colored, or refracted, decreases in receding from the brilliant point, while the number of colored rays determines the apparent clearness and strength of color, it would seem that the projection of a body would be more accurately shaded by a mixture of India ink with its conventional tint, than by shading it with ink alone, to express its gradations of shades, and then covering it with a flat tint of the conventional color. For the latter method would give so little color compara-

tively to the very dark shades, that it would be imperceptible, while in fact every part of the body does reveal its proper color to some extent. Otherwise, if the conventional flat tint were made strong enough to show, at the line of shade, it would obliterate the light ink-shading beneath it, in the vicinity of the brilliant point. In geometrical drawing, however, one of the exaggerations, which compensate for the artificial character of all projections, is secured by the second method of treating colored objects. Hence the view presented in the "Elementary Projection Drawing" (182).

b. There are three simple methods which may be used, separately or together, in executing a shaded drawing. These methods are: 1. By softened tints. 2. By superposed flat tints. 3. By touches.

c. Shading in softened tints. This is difficult to execute, but would give, when perfectly done, the most perfectly smooth and continuous gradation of shade. Moreover, it requires to be done at one operation, with a mixture of ink and the appropriate conventional tint, as explained in (*a*). For if the two shades be made separately, though both by this same method of softened tints, discrepancies between their gradations will be apt to spoil the intended effect of each.

d. Shading by superposed flat tints. This consists in placing a narrow dark tint on the line of shade; then, when dry, another, covering the first, and extending a little further, and so on. In this method, the drier the brush, the less will the edges of the successive bands show.

In the last method, we *rely somewhat on a free conformity* of the softened shades to their theoretically true distribution, as determined by observation merely.

In this method, we aim to have each of the well marked successive bands of tint exactly located so as to represent the bands of equal apparent brilliancy on the body represented. They should therefore, on account of their conspicuousness, be located *according to precise knowledge* of their proper position.

But this location it is difficult, if not impossible, to effect, by any simple or generally available construction, owing to the complication of the problem, even in the simplest cases, as indicated in (124 *b*).

e. Shading by touches. This method consists in placing directly upon each part of the drawing, a small quantity of ink,